The image of the Specht module under the inverse Schur functor

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Background: Specht modules and dual Weyl modules

Let K be an infinite field. Fix integers d and n.

Specht modules, S^λ

- representations of S_n
- indexed by partitions of n
- when $\operatorname{char} K = 0$, $\{ S^\lambda \mid \lambda \vdash n \}$ is a complete set of simple modules
- when $\operatorname{char} K = p$, $\{ D^\lambda \mid \lambda \vdash n,\ \lambda \text{ is } p\text{-regular} \}$ is a complete set of simple modules, where $D^\lambda = {}^{S^\lambda}\!/_{\operatorname{rad} S^\lambda}$

Dual Weyl modules, $\nabla^\lambda(E)$

- polynomial representations of $\mathrm{GL}_d(K)$ of degree n
- indexed by partitions of n
- when $\operatorname{char} K = 0$, $\{ \nabla^\lambda(E) \mid \lambda \vdash n \}$ is a complete set of simple modules
- when $\operatorname{char} K = p$, $\{ \operatorname{soc} \nabla^\lambda(E) \mid \lambda \vdash n \}$ is a complete set of simple modules

Background: Schur functor and inverse

Schur functor, $\mathcal{F}$

- $\{$polynomial repns of $\mathrm{GL}_d(K)$ of degree $n\} \to \{$repns of $S_n\}$
- defined for each $d \geqslant n$
- $\mathcal{F}(V)$ is the $(\underbrace{1,\ldots,1}_{n},0,\ldots,0)$-weight space of V

Inverse Schur functor, $\mathcal{G}_\otimes$

- $\{$repns of $S_n\} \to \{$polynomial repns of $\mathrm{GL}_d(K)$ of degree $n\}$
- defined for all d
- $\mathcal{G}_\otimes(U) = E^{\otimes n} \otimes_{KS_n} U$ where E is the natural representation of $\mathrm{GL}_d(K)$
- right-inverse to $\mathcal{F}$ (that is, $\mathcal{F}\mathcal{G}_\otimes(U) \cong U$)
- left-adjoint to $\mathcal{F}$

$\mathcal{F}$ also has a right-adjoint right-inverse, $\mathcal{G}_{\mathrm{Hom}}$, defined by

$$\mathcal{G}_{\mathrm{Hom}}(U) = \mathrm{Hom}_{KS_n}(E^{\otimes n}, U)$$

$\mathcal{G}_{\mathrm{Hom}}$ satisfies $\mathcal{G}_\otimes(-) = \mathcal{G}_{\mathrm{Hom}}(-^*)^\circ$

Problem: what is $\mathcal{G}_{\otimes}(S^{\lambda})$?

Known:

- $\mathcal{F}(\nabla^{\lambda}(E)) \cong S^{\lambda}$
- $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{\lambda}(E)$ when $p \geqslant 5$ [Kleschev–Nakano, 01]

Question:

- Does $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{\lambda}(E)$ hold when $p = 2$ or $p = 3$?

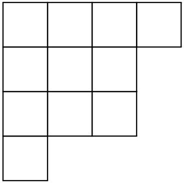
Answer:

- for $p = 3$: yes
- for $p = 2$: depends on λ

Young diagrams and tableaux

Partition $\lambda \vdash n$

Totally ordered set $\mathcal{B} \cong [d]$



Young diagram $[\lambda]$ of $\lambda = (4, 3, 3, 1)$

1	1	2	3
2	3	5	
4	5	6	
7			

Tableau of shape λ with entries in $\mathcal{B} = [7]$

Formally: map $[\lambda] \to \mathcal{B}$

Young diagrams and tableaux

Let G be a group, V a KG-module and $\mathcal{B}$ an ordered basis for V

G acts on tableaux entrywise, as if on $V^{\otimes n}$

Example:

Suppose $\mathcal{B} = \{v_1, v_2, v_3\}$, $t = \begin{array}{|c|c|c|}\hline v_1 & v_1 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array}$ and $g \in G$ is such that

$$g \cdot v_1 = v_1 + \alpha v_2, \quad g \cdot v_2 = v_2, \quad g \cdot v_3 = v_3$$

$$g \cdot t = \begin{array}{|c|c|c|}\hline g \cdot v_1 & g \cdot v_1 & g \cdot v_2 \\ \hline g \cdot v_2 & g \cdot v_3 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline v_1+\alpha v_2 & v_1+\alpha v_2 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array}$$

$$= \begin{array}{|c|c|c|}\hline v_1 & v_1 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array} + \alpha \begin{array}{|c|c|c|}\hline v_2 & v_1 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array} + \alpha \begin{array}{|c|c|c|}\hline v_1 & v_2 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array} + \alpha^2 \begin{array}{|c|c|c|}\hline v_2 & v_2 & v_2 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array}$$

Place permutation action

The symmetric group $S_{[\lambda]}$ acts on tableaux by *place permutation* (on the right)

Example:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \cdot \Big((1,1) \;\; (2,1) \;\; (2,2) \Big) = \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}$$

Formally:

$$(t \cdot \sigma)(b) = t(b\sigma^{-1})$$

for t a tableau, $\sigma \in S_{[\lambda]}$ and $b \in [\lambda]$

Column tabloids

$$\begin{array}{|c|c|c|c|} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & \\ 4 & 5 & 6 & \\ 7 & & & \end{array} \quad = \quad - \begin{array}{|c|c|c|c|} 4 & 1 & 2 & 3 \\ 2 & 3 & 5 & \\ 1 & 5 & 6 & \\ 7 & & & \end{array}$$

Formally: quotient $V^{\otimes n}$ by relations arising from signed column perms

BUT "relations arising from signed column perms" is ambiguous

(a) "alternating" relations: $\{\, x \mid x \cdot \sigma = -x \operatorname{sgn} \sigma \,\}$ $\rightsquigarrow$ $\bigwedge^{\lambda'} V$

(b) "skew" relations: $\{x \cdot \sigma - x \operatorname{sgn} \sigma\}$ $\rightsquigarrow$ $\mathrm{Sk}^{\lambda'} V$

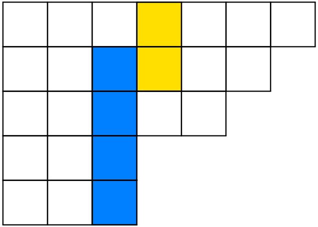
Tableau t $\rightsquigarrow$ alternating column tabloid $|t|$

$\rightsquigarrow$ skew column tabloid $||t||$

There is a surjection $q \colon \mathrm{Sk}^{\lambda'} V \twoheadrightarrow \bigwedge^{\lambda'} V$

$$||t|| \mapsto |t|$$

Garnir relations



Choose:

- two columns $j < j'$ of $[\lambda]$
- subsets $A \subseteq \mathrm{col}_j[\lambda]$ and $B \subseteq \mathrm{col}_{j'}[\lambda]$ such that $|A| + |B| > \lambda'_j$
- a set $\mathcal{S}$ of coset representatives for $S_A \times S_B$ in $S_{A \sqcup B}$

Given a tableau t of shape λ, the *Garnir relation* labelled by (t, A, B) is

$$G_{(t,A,B)} = \sum_{\tau \in \mathcal{S}} |t \cdot \tau| \operatorname{sgn} \tau$$

$\mathrm{GR}^\lambda(V)$ is the subspace of $\bigwedge^{\lambda'} V$ spanned by the Garnir relations

Presentations of S^λ and $\nabla^\lambda(E)$

Define $\nabla^\lambda(V)$ to be the quotient

$$\nabla^\lambda(V) \cong \bigwedge^{\lambda'} V \big/ \mathrm{GR}^\lambda(V)$$

with quotient map $\bigwedge^{\lambda'} V \twoheadrightarrow \nabla^\lambda(V)$ denoted e

Taking $V = E$ the natural representation of $\mathrm{GL}_d(K)$:

$$0 \longrightarrow \mathrm{GR}^\lambda(E) \longrightarrow \bigwedge^{\lambda'} E \xrightarrow{\ e\ } \nabla^\lambda(E) \longrightarrow 0$$

Taking $V = W$ the natural permutation representation of S_n:

restrict to tabloids *of symmetric type*

$$0 \longrightarrow \mathrm{GR}^\lambda_{\mathrm{sym}}(W) \longrightarrow \bigwedge^{\lambda'}_{\mathrm{sym}} W \xrightarrow{\ e\ } S^\lambda \longrightarrow 0$$

Applying $\mathcal{G}_\otimes$

$$\longrightarrow \mathcal{G}_\otimes(\mathsf{GR}^{\lambda}_{\mathrm{sym}}(W)) \xrightarrow{\mathcal{G}_\otimes(\iota)} \mathcal{G}_\otimes(\textstyle\bigwedge^{\lambda'}_{\mathrm{sym}} W) \xrightarrow{\mathcal{G}_\otimes(e)} \mathcal{G}_\otimes(S^\lambda) \longrightarrow 0$$

$$E^{\otimes n} \otimes_{KS_n} \textstyle\bigwedge^{\lambda'}_{\mathrm{sym}} W \;\cong\; \mathsf{Sk}^{\lambda'} E$$

e.g. $$x_1 \otimes \cdots \otimes x_5 \otimes_{KS_n} \begin{array}{|c|c|c|} 1 & 2 & 4 \\ 3 & 5 & \end{array} \leftrightarrow \begin{array}{|c|c|c|} x_1 & x_2 & x_4 \\ x_3 & x_5 & \end{array}$$

Kernel of $\mathcal{G}_\otimes(e)$ is $\operatorname{im} \mathcal{G}_\otimes(\iota)$, consisting of *skew Garnir relations*

$$\ker \mathcal{G}_\otimes(e) \cong \mathsf{SkGR}^\lambda(E)$$

$$0 \longrightarrow \mathsf{SkGR}^\lambda(E) \longrightarrow \mathsf{Sk}^{\lambda'} E \longrightarrow \mathcal{G}_\otimes(S^\lambda) \longrightarrow 0$$

Applying $\mathcal{G}_{\otimes}$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker q|_{\mathrm{GR}} & \hookrightarrow & \mathrm{SkGR}^{\lambda}(E) & \xrightarrow{q|_{\mathrm{GR}}} & \mathrm{GR}^{\lambda}(E) & \longrightarrow & 0 \\
 & & \big\downarrow\!\!\!\!\!\!^{\cap} & & \big\downarrow\!\!\!\!\!\!^{\cap} & & \big\downarrow\!\!\!\!\!\!^{\cap} & & \\
0 & \longrightarrow & \ker q & \hookrightarrow & \mathrm{Sk}^{\lambda'}E & \xrightarrow{q} & \bigwedge^{\lambda'} E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle e} & & \\
0 & \longrightarrow & {}^{\ker q}\!\big/_{\ker q|_{\mathrm{GR}}} & \longrightarrow & \mathcal{G}_{\otimes}(S^{\lambda}) & \longrightarrow & \nabla^{\lambda}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{\lambda}(E) \quad \Longleftrightarrow \quad \ker q \subseteq \mathrm{SkGR}^{\lambda}(E)$$

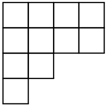
Characteristic 2

Theorem (McD. 20)

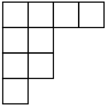
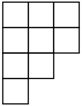
Suppose char $K = 2$ *and* $d \geqslant n - 2$.
There is an isomorphism $\mathcal{G}_{\otimes}(S^{\lambda}) \cong \nabla^{\lambda} E$ *if and only if* λ *is 2-regular, or* $\lambda_1 = \lambda_2 \geqslant \lambda_3 + 2$ *and* λ *minus its first part is 2-regular.*

Examples:

$$\mathcal{G}_{\otimes} S^{\lambda} \cong \nabla^{\lambda}(E)$$

- λ 2-regular
- $\lambda =$ 

$$\mathcal{G}_{\otimes} S^{\lambda} \not\cong \nabla^{\lambda}(E)$$

- $\lambda =$ 
- $\lambda =$ 

Example: two-row partitions

Aim: show $\ker q \subseteq \mathrm{SkGR}^{\lambda}(E)$ when λ has only two rows $(\lambda \neq (1,1))$

$\ker q = \langle$skew column tabloids with a repeat in a column$\rangle_K$

Suppose $||t|| \in \ker q$

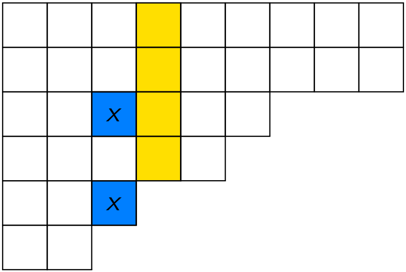
Write $t = \begin{array}{|c|c|c|c|} \hline * & x & y & * \\ \hline * & x & * \\ \cline{1-3} \end{array}$

Choose A to be a pair of boxes containing repeats in a column, and B any other box

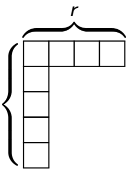
$$\begin{aligned} G^{\mathrm{sk}}_{(t,A,B)} &= \left|\begin{array}{c|c|c|c} * & x & y & * \\ * & x & * & \end{array}\right| + \left|\begin{array}{c|c|c|c} * & y & x & * \\ * & x & * & \end{array}\right| + \left|\begin{array}{c|c|c|c} * & x & x & * \\ * & y & * & \end{array}\right| \\ &= \left|\begin{array}{c|c|c|c} * & x & y & * \\ * & x & * & \end{array}\right| \\ &= ||t|| \end{aligned}$$

Showing $\ker q \subseteq \mathrm{SkGR}^{\lambda}(E)$

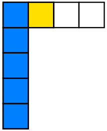
Consider the right-most column in which there is a repeated entry



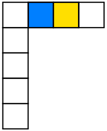
Example: hook partitions

Suppose $\lambda = (r, 1^{s-1}) = s$  (with $s \geqslant 3$, $r \geqslant 2$)

Two types of Garnir relations:



($\rightsquigarrow$ $s+1$ summands)

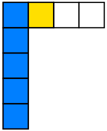


($\rightsquigarrow$ 2 summands)

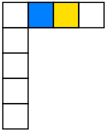
Can we find a skew column tabloid $||t|| \in \ker q$ that cannot be written as a linear combination of skew Garnir relations?

Example: hook partitions

Consider the tableau t with all entries 1



$$G^{\mathrm{sk}}_{(t,A,B)} = (s+1)||t|| = \begin{cases} ||t|| & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd} \end{cases}$$



$$G^{\mathrm{sk}}_{(t,A,B)} = 2||t||0$$

So $\ker q \not\subseteq \mathrm{SkGR}^{\lambda}(E)$ when s is odd

Remarks

Can deduce that S^λ is indecomposable whenever $\mathcal{G}_\otimes(S^\lambda) \cong \nabla^\lambda(E)$ (which is more often than just λ is 2-regular)

When $\mathcal{G}_\otimes(S^\lambda) \ncong \nabla^\lambda(E)$:

- $\nabla^\lambda(E)$ is still a quotient of $\mathcal{G}_\otimes(S^\lambda)$ via
$$0 \longrightarrow {}^{\ker q}\!/\!_{\ker q|_{\mathrm{GR}}} \longrightarrow \mathcal{G}_\otimes(S^\lambda) \longrightarrow \nabla^\lambda(E) \longrightarrow 0$$
- the kernel has no 2-restricted composition factors
- the dimension of the kernel is small compared to $\nabla^\lambda(E)$ ($O(d^{n-1})$ rather than $O(d^n)$ as d varies)
- the kernel cannot have a ∇-filtration
- $\mathcal{G}_\otimes(S^\lambda)$ need not have a ∇-filtration (e.g. $\lambda = (2,2,1)$)

Thank you