Decomposition numbers for unipotent blocks with small $\mathfrak{sl}_2$-weight in finite classical groups

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Based on joint work with Olivier Dudas, to appear soon.

Dream of categorification

$\mathfrak{g}$ a Lie algebra (finite or affine Dynkin type)

$\mathcal{C}$ an abelian category, finite-length

Definition (Chuang-Rouquier)

A $\mathfrak{g}$-**categorification on** $\mathcal{C}$ is a collection of exact endofunctors $\{E_i, F_i\}$ of $\mathcal{C}$, where i ranges over the nodes of the Dynkin diagram of $\mathfrak{g}$, satisfying:

- For each i, E_i and F_i are a biadjoint pair of functors;
- The functors E_i and F_i for all i induce an action of $\mathfrak{g}$ on the (complexified) Grothendieck group $[\mathcal{C}]$ via $[E_i] = e_i$, $[F_i] = f_i$ where e_i, f_i are the Chevalley generators of $\mathfrak{g}$;
- The classes $[S]$ in $[\mathcal{C}]$ of the simple objects $S \in \mathcal{C}$ are $\mathfrak{g}$-weight vectors;
- Strong: Set $E = \bigoplus_i E_i$, $F = \bigoplus_i F_i$. There are natural transformations $X \in \mathrm{End}(F)$ and $T \in \mathrm{End}(F^2)$ such that in $\mathrm{End}(F^n)$, $X_j := 1^{j-1} X 1^{n-j}$ and $T_k := 1^{k-1} T 1^{n-k-1}$ satisfy defining relations of an affine Hecke algebra.

Often: $\mathcal{C}$ is a tower of module categories, E and F are restriction and induction functors.

Example: the symmetric groups (Chuang-Rouquier, LLT, Kleshchev)

Let $\mathrm{char}\ k = p > 0$ and consider $\mathcal{C} = \bigoplus_{n \geq 0} kS_n\text{-mod}$.

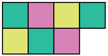
Theorem (Chuang-Rouquier, Lascoux-Leclerc-Thibon)

There is a $\widehat{\mathfrak{sl}}_p$-categorification on $\mathcal{C}$ with

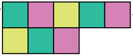
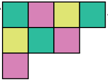
$$\mathrm{Res} = E = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} E_i, \qquad \mathrm{Ind} = F = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} F_i$$

If Δ_λ, $\lambda \vdash n$, is a Specht module, then

$$[E_i(\Delta_\lambda)] = \sum_{\substack{b \in \mathrm{Remov}(\lambda) \\ \mathrm{ct}(b) \equiv i \mod p}} [\Delta_{\lambda \setminus b}], \qquad [F_i(\Delta_\lambda)] = \sum_{\substack{b \in \mathrm{Add}(\lambda) \\ \mathrm{ct}(b) \equiv i \mod p}} [\Delta_{\lambda \cup b}]$$

Illustration: $p = 3$, $\lambda =$ 

$[E_1(\Delta$  $)] = [\Delta$  $]$,

$[F_1(\Delta$  $)] = [\Delta$  $] + [\Delta$  $]$

The symmetric groups, continued

$\lambda \vdash n$ a p-regular partition, S_λ a simple kS_n-module. Fix $i \in \mathbb{Z}/p\mathbb{Z}$.

If $F_i(S_\lambda) \neq 0$ then:

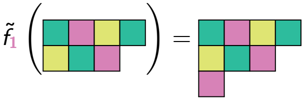
- $F_i(S_\lambda)$ is indecomposable,
- $F_i(S_\lambda)$ has simple head and socle,
- $\mathrm{head}(F_i(S_\lambda)) \cong \mathrm{socle}(F_i(S_\lambda))$

Define $S_{\tilde{f}_i(\lambda)}$ by $F_i(S_\lambda) \twoheadrightarrow S_{\tilde{f}_i(\lambda)}$

Combinatorial rule for finding $\tilde{f}_i(\lambda)$: for $i \in \mathbb{Z}/p\mathbb{Z}$,

$$\tilde{f}_i(\lambda) = \lambda \cup \{\text{“good addable i-box”}\}$$

Example ($p = 3$):



Focus on $\mathfrak{sl}_2$

Given a $\mathfrak{g}$-categorification on $\mathcal{C}$, each pair (E_i, F_i) generates an $\mathfrak{sl}_2$-categorification on $\mathcal{C}$. Study one $\mathfrak{sl}_2$-categorification at a time.

Fix $\mathfrak{g} = \mathfrak{sl}_2$, have $\mathfrak{sl}_2$-categorification:

$$\mathcal{C} = \bigoplus_{\omega \in \mathbb{Z}} \mathcal{C}_\omega$$

where the $\mathcal{C}_\omega$ are *weight categories*. Exact, biadjoint functors E, F shift weights by ± 2:

$$\mathcal{C}_\omega \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{C}_{\omega+2}$$

and $[E][F] - [F][E]$ acts by multiplication by ω on $[\mathcal{C}_\omega]$.

Divided power operators $E^{(n)}$ and $F^{(n)}$ satisfy

$$E^n \simeq \left(E^{(n)}\right)^{\oplus n!} \quad \text{and} \quad F^n \simeq \left(F^{(n)}\right)^{\oplus n!}.$$

Functors E, F on simple modules

R a ring, $\mathcal{C}$ an R-linear abelian category, finite length, with $\mathfrak{sl}_2$-categorification.
Assume $\mathrm{End}(S) \cong R$ for every simple object $S \in \mathcal{C}$.

- If $E(S) \neq 0$ then $E(S)$ has simple head and simple socle, and
- $\mathrm{head}(E(S)) \cong \mathrm{socle}(E(S))$.

Extend to the divided power functors:

Lemma (Chuang-Rouquier)

Let $S \in \operatorname{Irr} \mathcal{C}_\omega$ and $n \geq 0$ be such that $E^{n+1}(S) = 0$ and $E^n(S) \neq 0$.

1. *$E^{(n)}(S)$ is simple.*
2. *The socle and head of $F^{(n)}E^{(n)}(S)$ are isomorphic to S.*
3. *The simple module S occurs in $F^{(n)}E^{(n)}(S)$ with multiplicity $\binom{\omega+2n}{n}$ as a composition factor.*

Decomposition numbers

$\mathcal{O}$ a complete DVR with residue field k, fraction field K; char $k = \ell > 0$, char $K = 0$.

Let $\{G_r\}_{r\in\mathbb{N}}$ be a family of finite groups, $\Lambda \in \{\mathcal{O}, k, K\}$,

$$\Lambda\mathcal{G} = \bigoplus_{r\geq 0} \Lambda G_r\text{-mod}.$$

Assume k and K are "large enough." Then:

- Every $S \in \mathrm{Irr}_k\mathcal{G}$ has a projective cover P_S in $k\mathcal{G}$, unique up to isomorphism.
- Every projective module P in $k\mathcal{G}$ lifts uniquely to a projective module $\widetilde{P}$ in $\mathcal{O}\mathcal{G}$.
- $K\mathcal{G}$ is semisimple.
- $k\mathcal{G}$ has finite length and $\mathrm{End}(S) \cong k$ for all simples $S \in \mathrm{Irr}_k\mathcal{G}$.

Let $S \in \mathrm{Irr}_k\mathcal{G}$ and $\Delta \in \mathrm{Irr}_K\mathcal{G}$. The **decomposition number**

$$[P_S : \Delta]$$

is the multiplicity of Δ as a direct summand of $K \otimes_{\mathcal{O}} \widetilde{P_S}$.

Finite classical groups and modular representation theory

G_n a finite classical group, one of

$$\mathrm{SO}_{2n+1}(q), \mathrm{Sp}_{2n}(q), \mathrm{O}^+_{2n}(q), \mathrm{O}^-_{2n}(q),$$

so Weyl group of G_n is B_n or D_n.

$|\mathrm{Irr}_K(G_n)|$ depends on q, however $\mathrm{Irr}_K(G_n)$ contains a subset of **unipotent representations** indexed by elements of various Weyl groups of types B and D independent of q.

Fix an ℓ-modular system $(\mathcal{O}, k, K)$, with k and K large enough.
$\mathrm{char}\, k = \ell > 0$, $|q| = d \mod \ell$, $d \geq 2$ even: "unitary prime case."

The "quantum characteristic" d plays the role that characteristic p did for kS_n. Analogous to Hecke algebra at a d'th root of 1.

$\Delta \in \mathrm{Irr}_K(G_n)$ unipotent, $S \in \mathrm{Irr}_k G$.

Problem (open)

Describe the decomposition numbers $[P_S : \Delta]$ *of* G_n.

$\widehat{\mathfrak{sl}}_d$-action on the unipotent category of G_n

$\Lambda = k$ or K
$\mathcal{C}_n = \Lambda G_n^{\mathrm{unip}}$ the sum of those blocks of ΛG_n containing a unipotent representation
$\mathcal{C} = \bigoplus_{n\geq 0} \mathcal{C}_n$

$\mathrm{Res}^n_{n-1} : \mathcal{C}_n \to \mathcal{C}_{n-1}$ Harish-Chandra restriction,
$\mathrm{Ind}^n_{n-1} : \mathcal{C}_{n-1} \to \mathcal{C}_n$ Harish-Chandra induction.

$\mathrm{Res} = \bigoplus_n \mathrm{Res}^n_{n-1}$, $\mathrm{Ind} = \bigoplus_n \mathrm{Ind}^n_{n-1}$ are exact, biadjoint endofunctors of $\mathcal{C}$.

Theorem (Dudas-Varagnolo-Vasserot (type B)+others (type D, in progress))

There is a $\widehat{\mathfrak{sl}}_d$-categorification on $\mathcal{C}$ with $\mathrm{Res} = E$ and $\mathrm{Ind} = F$.

$$\mathrm{Res} \cong \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} E_i, \qquad \mathrm{Ind} \cong \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} F_i$$

As with kS_n, have combinatorial recipes for action of E_i, F_i on Δ, S.

Combinatorics of unipotent representations

Unipotent representations $\Delta \in \mathrm{Irr}_K(G_n)$ are labeled by **symbols**.

A symbol is a **charged bipartition** $|\lambda^1.\lambda^2, (\sigma_1, \sigma_2)\rangle$ presented as a 2-row abacus.

Example

The following symbol labels a unipotent character of type B:

row 2	●	●	●	●	●	●	●	●	·	●	·	●	●	·
row 1	●	·	●	●	●	·	·	·	·	·	·	·	·	·
	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6

The bipartition is $\lambda^1.\lambda^2 = (1^3).(2^2, 1)$. The charge is $(\sigma_1, \sigma_2) = (-4, 3)$. We recover the Harish-Chandra series of the unipotent character from (σ_1, σ_2). We have $\Delta = B_{3^2+3} : \lambda^1.\lambda^2$.

Addable i-box:

- **row 2:** bead in position $i + \frac{d}{2} \mod d$, space in position $i+1$,
- **row 1:** bead in position $i \mod d$, space in position $i+1$.

F_i acts on unipotent Δ analogously to symmetric group case: add all possible i-boxes.

When does the categorification control decomposition numbers?

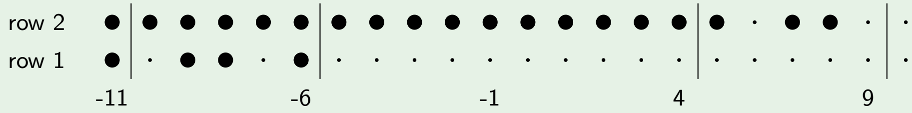
The $\widehat{\mathfrak{sl}}_d$-categorification comes from induction and restriction, so it should not know decomposition numbers $[P_S : \Delta]$ if $S \in \mathrm{Irr}_k(G_n)$ is cuspidal.

But we can use it to understand chunks of the decomposition matrix of G_n. This will be square submatrices with rows and columns labeled by "d-small symbols" with the same charge.

Rather than define d-small, an example.

Example

Let $d = 10$. The following symbol is d-small:



The **left region**, **middle region**, and **right region** are indicated by the bars. The left and right regions each have length $\frac{d}{2}$ and are relatively positioned by a shift of $\frac{d}{2}$ mod d.

The up-down diagram of a d-small symbol

We will associate to d-small symbols some combinatorics due to Brundan and Stroppel.

Θ a d-small symbol

Define the *up-down diagram* of a d-small symbol:
$\mathrm{w}_{\wedge\vee}(\Theta) = w_1 w_2 \dots w_{\frac{d}{2}}$,
where $w_i \in \{\wedge, \vee, \circ, \times\}$ for each $i = 1, \dots, \frac{d}{2}$.

$\mathrm{w}_{\wedge\vee}(\Theta)$ is determined by the left and right regions of Θ.

The i'th letter w_i records what happens in the i'th position in the left and right regions:

- if the right region has a bead and the left region has two beads then $w_i = \times$,
- if the right region has a bead and the left region has one bead, then $w_i = \wedge$,
- if the right region has no bead and the left region has two beads, then $w_i = \vee$,
- if the right region has no bead and the left region has one bead, then $w_i = \circ$.

Example

Continuing with the symbol from the previous example, we have $\mathrm{w}_{\wedge\vee}(\Theta) = \wedge \vee \times \wedge \vee$.

The cup diagram of a d-small symbol

Form the cup diagram $c_{\wedge\vee}(\Theta)$ of $w_{\wedge\vee}(\Theta)$ by attaching counter-clockwise oriented arcs ("cups") and rays below the $\wedge$'s and $\vee$'s of $w_{\wedge\vee}(\Theta)$.

Do this recursively:

- attach a cup to adjacent $\vee\wedge$ or $\vee \ldots \wedge$ where $\ldots$ only contains $\times$ and $\circ$ symbols,
- attach a cup to $\vee \ldots \wedge$ where $\ldots$ contains only $\times$, $\circ$, or previously constructed cups,
- when no more cups can be attached, attach rays to remaining $\wedge$ and $\vee$ symbols.

Example

For $w_{\wedge\vee}(\Theta) = \wedge \vee \times \wedge \vee$ we get

$c_{\wedge\vee}(\Theta) =$ 

Theorem

Our main theorem is a closed combinatorial formula for the square submatrix of the decomposition matrix cut out by a *d*-small Harish-Chandra series within a block.

Theorem (Dudas-N. '23)

Suppose Θ, Θ' are d-small symbols with the same charge and describing bipartitions of the same size. Then

$$[P_\Theta : \Delta_{\Theta'}] = \begin{cases} 1 \text{ if } \mathrm{w}_{\wedge\vee}(\Theta') \text{ is obtained from } \mathrm{w}_{\wedge\vee}(\Theta) \text{ by reversing the orientation} \\ \quad \text{on a subset of the cups of } \mathrm{c}_{\wedge\vee}(\lambda), \\ 0 \text{ otherwise.} \end{cases}$$

This is the same formula given by Brundan and Stroppel for the decomposition numbers of the extended Khovanov arc algebra $K_{m,n}$ where $m = \#\{\wedge\}$ and $n = \#\{\vee\}$. From results of Stroppel it follows that:

Corollary

With the same assumptions on Θ, Θ', the decomposition number $[P_\Theta : \Delta_{\Theta'}]$ is given by a parabolic Kazhdan-Lusztig polynomial $p_{\lambda,\mu}(t)$ of type $S_m \times S_n \subset S_{m+n}$ evaluated at $t = 1$. The partitions λ, μ may be read off of $\mathrm{w}_{\wedge\vee}(\Theta), \mathrm{w}_{\wedge\vee}(\Theta')$.

How do we get such a formula?

Restriction preserves d-small symbols: if Θ is d-small then $\tilde{e}_i(\Theta)$, $e_i(\Theta)$ are again d-small, or 0.

Moreover, a d-small symbol has at most 2 total addable and removable i-boxes.

It follows that fixing any $i \in \mathbb{Z}/d\mathbb{Z}$, the d-small symbols belong to irreducible $\mathfrak{sl}_2$-representations of highest weight 0, 1, or 2 for each pair (E_i, F_i).

We show that belonging to an irreducible $\mathfrak{sl}_2$-representation of highest weight at most 2 determines the decomposition numbers $[P_\Theta : \Delta_{\Theta'}]$ by induction using some E_i, unless S_Θ is cuspidal.

We can easily classify cuspidals S_Θ if Θ is d-small, and it is trivial to understand $[P_\Theta : \Delta_{\Theta'}]$ in that case.

The effect of the i-induction F_i can be described directly on $\mathrm{w}_{\wedge\vee}(\Theta)$ and $\mathrm{c}_{\wedge\vee}(\Theta)$. This allows us to prove the formula for decomposition numbers inductively.

Some motivation for the result

The d-small symbols are constructed so that "it's as if $d = \infty$" from the perspective of the E_i's acting on simples and Δ's.

In the case $d = \infty$, Brundan and Stroppel showed that the blocks of the Hecke algebra of B_n are equivalent to $K_{k,m}$-mod for various k, m. The Hecke algebra of B_n has Specht (standard) and simple modules labeled by charged bipartitions, that is by symbols. Our construction of $\mathrm{w}_{\wedge\vee}(\Theta)$ is an adaptation of Brundan-Stroppel's construction.

It is known that the decomposition matrix of a Hecke algebra of type B embeds in the decomposition matrix of G_n for each Harish-Chandra series (excepting the principal series in types D_n and ${}^2D_n$ which give rise to Hecke algebras of type D, but principal series aren't d-small).

For d large enough relative to the size of bipartitions, the Hecke algebra decomposition matrix will be given by the same rules as in the $d = \infty$ case. Our result extends this to the whole square submatrix of the d-small Harish-Chandra series in its block.

Obligatory example to finish

The matrix of parabolic Kazhdan-Lusztig polynomials of type $(W, P) = (S_4, S_2 \times S_2)$, evaluated at 1...

The decomposition matrix of the extended Khovanov arc algebra $K_{2,2}$...

The submatrix of the decomposition matrix of $\mathrm{Sp}_{40}(q)$ given by the B_{3^2+3}-series in the block of $B_{3^2+3} : (1^3).(2^2, 1)$ when $d = 10$ ...

$$\begin{matrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & 1 & & 1 & & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & & & & 1 & 1 \end{matrix}$$