Computation of decomposition numbers of $\mathfrak{S}_n$

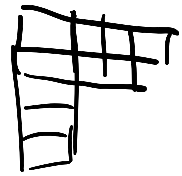
$\mathcal{H}_n = \mathcal{H}_{q,\mathbb{F}}(\mathfrak{S}_n)$, $q \in \mathbb{F}^*$, $q = 1$

e = mult order of q.

Main open problem: D^μ simple modules

μ e-reg partition of n,
↳ no e equal parts

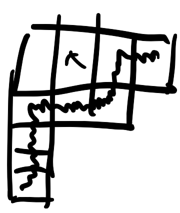
Ex: $\lambda = (4,3,1,1,1) \vdash 10$



$[S^\lambda : D^\mu] \geq 0$
↳ Specht mod

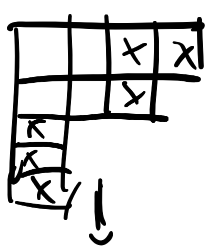
(1) $[S^\lambda : D^\mu]^e_{\mathbb{F}} > 0$ only if λ, μ belong to same block

- Def: - rim $(i,j) \in \lambda$ s.t. $(i+1, j+1) \notin \lambda$

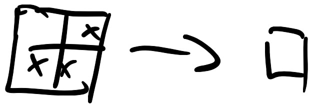


- rim e-hooks:

$e = 3$



- e-weight:



3-weight of $(4,3,1,1,1)$ is 3

$\Rightarrow$ e-core $\quad \rho \vdash n - we \Rightarrow$ 3-core of $(4,3,1,1,1)$
is $\rho = (1)$

thm: (Nakayama conj): $\lambda, \mu \vdash n$ belong to the same
(Brauer-Robinson thm)

block $\Leftrightarrow$ they have same e-core

- We can study decom numbers by block (by e-core)
- e-weight measures difficulty of rep theory of the block.

- Decomp number problem solved for $w \leq 2$
  - $w = 0$
  - $w = 1$
  - $w = 2$ Richards $p = \mathrm{Char}(\mathbb{F}) > 2$, Fayers, $p = 2$.
  - $w \geq 3$

(2) LLT algorithm when $\mathbb{F} = \mathbb{C}$

$$D = \left([S^\lambda : D^\mu] \right)_{(\lambda,\mu)}, \quad D_{\mathbb{F}} = D_{\mathbb{C}} \underset{\text{Adjustment matrix}}{A}$$

James's conj: If $w < \mathrm{char}(\mathbb{F}) \Rightarrow A = I$

(G. Williamson '17 found a counterexample)

James's conj holds for $w \leq 4$, principal block of $\mathcal{H}_{se}$ (Law).

3 $\mu, \lambda \vdash n$, $\mu \trianglerighteq \lambda$, $\forall j \ \sum_{i=0}^{j} \mu_i \geq \sum_{i=0}^{j} \lambda_i$

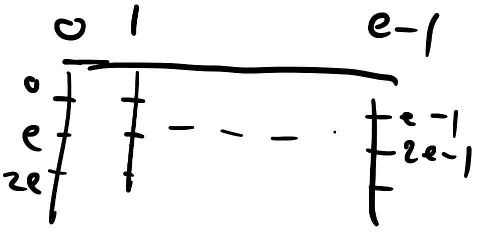
$\lambda = (4,3,1,1,1)$, $\mu = (3,3,2,2)$

thm: $[S^\lambda : D^\mu] = 0$ unless $\mu \trianglerighteq \lambda$

the abacus

Fix e, $r \geq s$

$\lambda = (\lambda_1, \ldots, \lambda_s)$

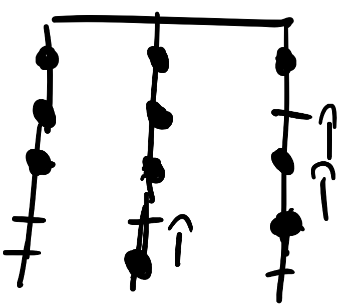


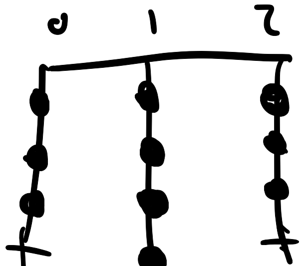
$r-s$ beads in the $r-s$ first positions

$\lambda_i - \lambda_{i+1}$ empty spaces between each bead

Ex: $\lambda = (4,3,1,1,1)$, $e=3$, $r=10$

$r-s=5$



$\rho =$ 

$= (1)$

(4) Block B
runner i there are k more beads than in runner $i-1$ either;

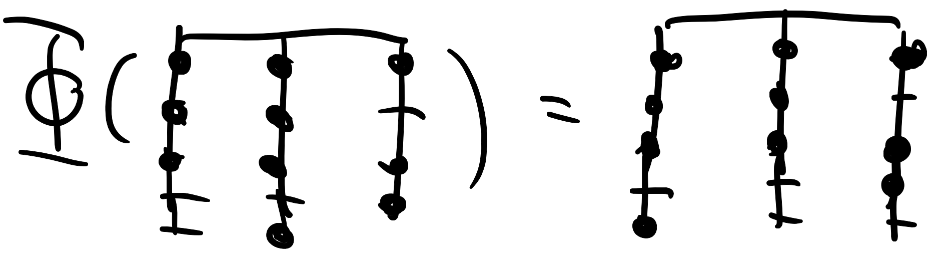
- If $k \geq w \Rightarrow \Phi : B \twoheadrightarrow \hat{B}$ s.t.

$$[S^\lambda : D^\mu] = [S^{\Phi(\lambda)} : D^{\Phi(\mu)}]$$

$\Phi(\lambda)$ build from λ by swapping runners $i, i-1$.

- If $k < w \Rightarrow$ If in runner i in λ there are more than k beads with empty space to their left, then λ exceptional

Ex: $[S^{(4,3,1,1,1)} : D^\mu] = [S^{(3,3,1,1,1)} : D^{\Phi(\mu)}]$

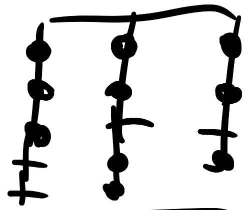
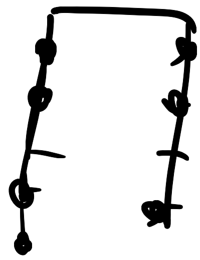


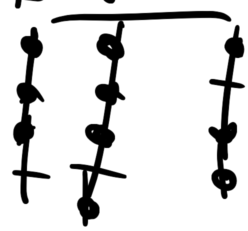
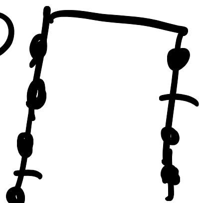
$= (3,3,1,1,1)$

(5) $\lambda, \mu \in B$, runner i define a function

$\mathcal{L}_i(\lambda) = \mathcal{L}_i(\mu)$ and λ, μ i-empty

$$\Rightarrow [S^\lambda : D^\mu]^e = [S^{\lambda^{-i}} : D^{\mu^{-i}}]^{e-1}$$

Ex: $\mu = (4,3,3) =$  $\longrightarrow$  $= (2,2,2)$

$\lambda = (4,3,1,1,1) =$  $\longrightarrow$  $= (2,2,1,1)$

$$[S^\lambda : D^\mu]^3 = [S^{(2,2,1,1)} : D^{(2,2,2)}]^2$$

- M. Rosas max comp length principal block of $\mathbb{F}\Gamma_{3p}$ is 14. $(w=3)$