Spin representations of the symmetric group

Dean Yates

5th April 2022

Basic definitions

A projective representation is a group homomorphism into $PGL(V)$.

Schur: Projective representations of S_n are equivalent to linear representations of the group $\tilde{S}_n := \langle z, t_1, \ldots, t_{n-1} \rangle$ with relations $z^2 = 1$; $t_i^2 = z$ for each i; $(t_i t_{i+1})^3 = z$ for $i \leq n-2$; $(t_i t_j)^2 = z$ when $|i - j| > 1$.

$$S_n \cong \tilde{S}_n / \{1, z\}.$$

A *spin representation* of S_n is a linear representation of $\tilde{S}_n$ which maps $z \mapsto -I$.

$$\tilde{S}_n \to S_n \to \mathcal{P}_n$$

Schur: elements corresponding to $\lambda \in \mathcal{P}_n$ split into two conjugacy classes in $\tilde{S}_n$ *iff.* λ has no two equal parts.

$$\mathcal{B}_n := \{\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_n : \lambda_i = \lambda_j \Rightarrow i = j\};\ \mathcal{B} := \cup_{n \in \mathbb{N}} \mathcal{B}_n.$$

$[\lambda]$ is the spin representation associated to λ.

Schur classified the irreducible characters of $\tilde{S}_n$,
constructed faithful matrix representation for the *basic representation* $[(n)]$, in which t_i act via matrices $T_1, \dots, T_{n-1}$ satisfying

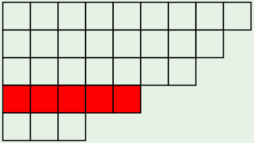
$$T_i^2 = I, \qquad T_i T_j = -T_j T_i \text{ for } |i - j| > 1,$$
$$(\Delta) \qquad T_i T_{i+1} + T_{i+1} T_i + I = 0.$$

For odd integers $p \geq 3$, removing a p-bar from λ can mean

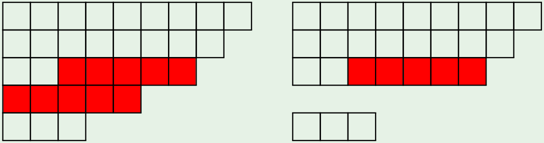
1. removing p from λ
2. replacing $x \in \lambda$ with $x - p$, when $0 < x - p \notin \lambda$
3. removing two parts $x, p - x \in \lambda$

If λ has no removable p-bars, we say that λ is a p-**bar-core**, and we denote the set of p-bar-cores by C_p

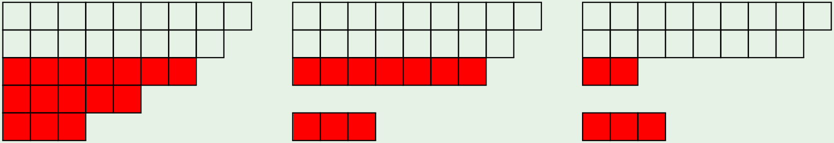
Example



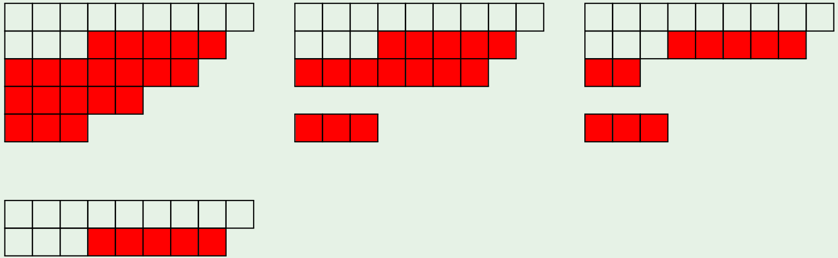
Example



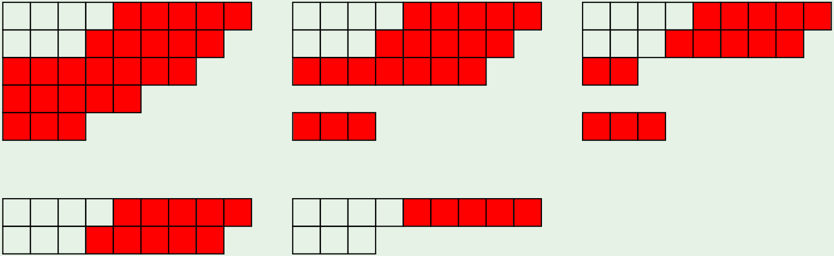
Example



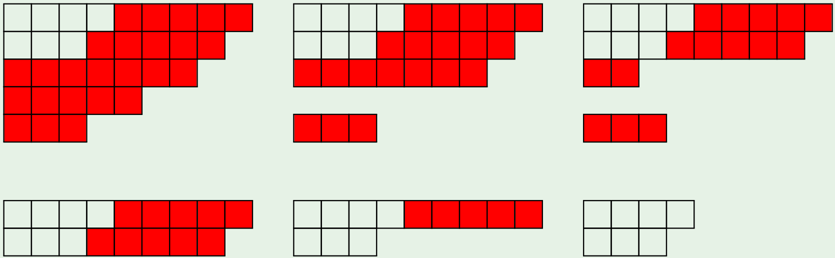
Example



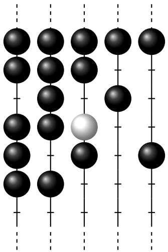
Example



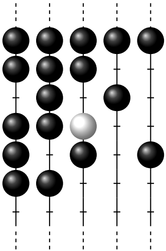
Example



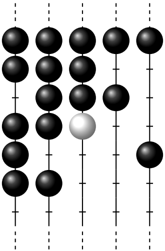
$(9, 8, 7, 5, 3)$



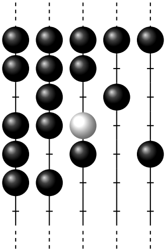
$(9, 8, 7, 5, 3)$



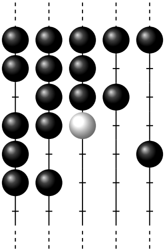
$(9, 8, 7, 3)$



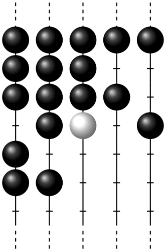
$(9, 8, 7, 5, 3)$



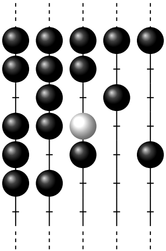
$(9, 8, 7, 3)$



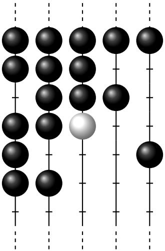
$(9, 8, 3, 2)$



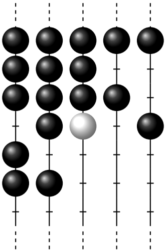
$(9,8,7,5,3)$

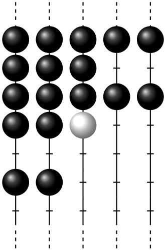


$(9,8,7,3)$



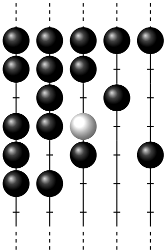
$(9,8,3,2)$



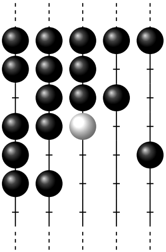


$(9,8)$

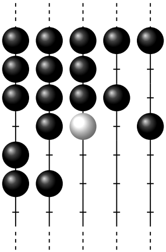
$(9, 8, 7, 5, 3)$

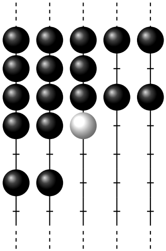


$(9, 8, 7, 3)$

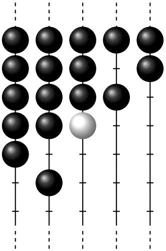


$(9, 8, 3, 2)$



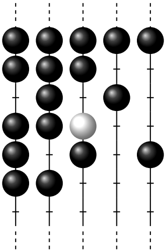


$(9, 8)$

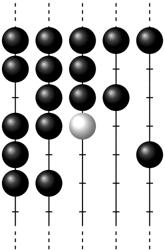


$(9, 3)$

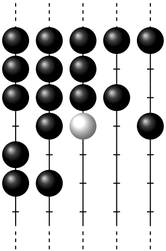
$(9, 8, 7, 5, 3)$

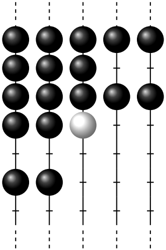


$(9, 8, 7, 3)$

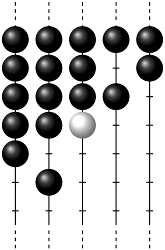


$(9, 8, 3, 2)$

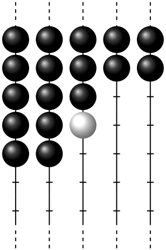




$(9, 8)$



$(9, 3)$



$(4, 3)$

$\overline{\lambda}_p$ is the unique p-bar-core of $\lambda \in \mathcal{B}$;

$$\overline{\mathrm{wt}}_p(\lambda) := \tfrac{|\lambda| - |\overline{\lambda}_p|}{p}$$

is the p-bar-weight of λ (where $|\lambda|$ is the sum of the parts of λ).

Example

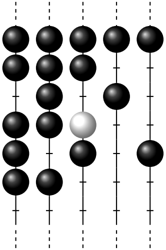
$\overline{(9,8,7,5,3)}_5 = (4,3)$, $\overline{\mathrm{wt}}_5((9,8,7,5,3)) = 5$

$$\mathcal{A}(\lambda) := \{x \in \mathbb{Z} | x \in \lambda \text{ or } x < 0, -x \notin \lambda\}$$

$$\mathcal{Q}_p(\lambda) := (\lambda^{(0 \text{ mod } p)}, \ldots, \lambda^{(p-1 \text{ mod } p)})$$

is the p-**quotient** of λ, where $\lambda^{(0 \text{ mod } p)} = \{x/p | x \in \lambda, x \in p\mathbb{Z}\}$, for $j \not\equiv 0 \pmod{p}$, the i^{th} part of the (not necessarily strict) partition $\lambda^{(j \text{ mod } p)}$ is equal to the number of empty spaces above the i^{th} lowest bead on runner j in the bead configuration for λ on the p-runner abacus.

$\lambda = (9, 8, 7, 5, 3)$ $\qquad$ $\mathcal{Q}_5(\lambda) = ((1), (1), (3), (1, 1, 1), (1))$



A level q group action on bar partitions

Let $\mathfrak{W}_p$ be the affine Coxeter group of type $\tilde{C}_{(p-1)/2}$, with generators $\delta_0, \delta_1 \ldots, \delta_{(p-1)/2}$, and relations

$$
\begin{aligned}
\delta_i^2 &= 1 && \text{for } 0 \leq i \leq \tfrac{p-1}{2} \\
\delta_i\delta_j &= \delta_j\delta_i && \text{when } 0 \leq i < j-1 \leq \tfrac{p-3}{2} \\
\delta_i\delta_{i+1}\delta_i &= \delta_{i+1}\delta_i\delta_{i+1} && \text{for } 1 \leq i \leq \tfrac{p-5}{2} \\
\delta_0\delta_1\delta_0\delta_1 &= \delta_1\delta_0\delta_1\delta_0 && \text{if } p > 3 \\
\delta_{\frac{p-3}{2}}\delta_{\frac{p-1}{2}}\delta_{\frac{p-3}{2}}\delta_{\frac{p-1}{2}} &= \delta_{\frac{p-1}{2}}\delta_{\frac{p-3}{2}}\delta_{\frac{p-1}{2}}\delta_{\frac{p-3}{2}} && \text{if } p > 3
\end{aligned}
$$

For coprime odd integers $p, q \geq 3$, we define a *level q action* of $\mathfrak{W}_p$ on $\mathbb{Z}$:

$$\delta_0 x = \begin{cases} x - 2q & \text{if } x \equiv q \pmod{p} \\ x + 2q & \text{if } x \equiv -q \pmod{p} \\ x & \text{otherwise} \end{cases}$$

$$\delta_i x = \begin{cases} x - q & \text{if } x \equiv (i+1)q, -iq \pmod{p} \\ x + q & \text{if } x \equiv iq, -(i+1)q \pmod{p} \\ x & \text{otherwise} \end{cases} \qquad (\text{for } 1 \leq i \leq \tfrac{p-1}{2})$$

We can extend this action from $\mathbb{Z}$ to $\mathcal{B}$ by acting on the set $\mathcal{A}(\lambda) := \{x \in \mathbb{Z} | x \in \lambda \text{ or } x < 0, -x \notin \lambda\}$ and defining $\delta_i \lambda$ to be the bar partition with $\mathcal{A}(\delta_i \lambda) = \delta_i \mathcal{A}(\lambda)$

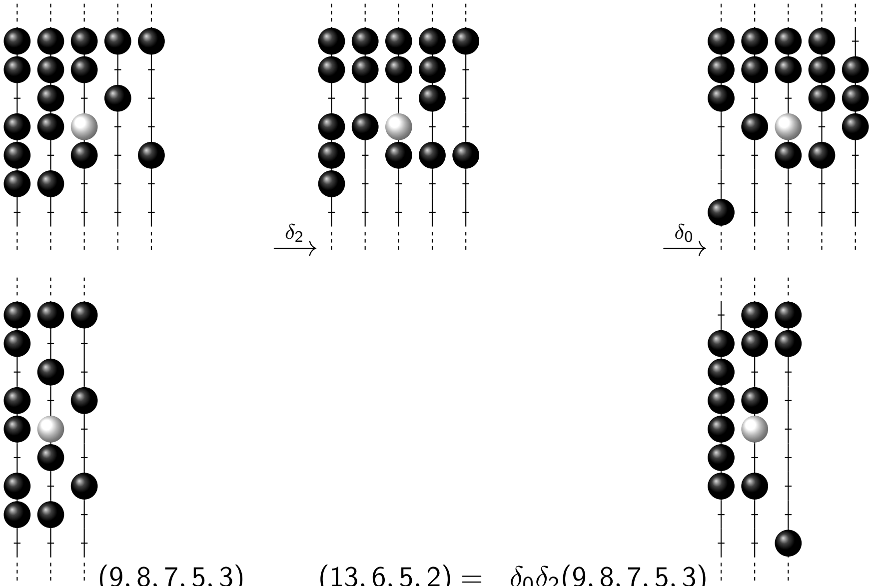
Example

Level 3 action of $\mathfrak{W}_5 = \langle \delta_0, \delta_1, \delta_2 \rangle$

$$\delta_0 x = \begin{cases} x-6 & \text{if } x \equiv 3 \pmod 5 \\ x+6 & \text{if } x \equiv 2 \pmod 5 \\ x & \text{otherwise} \end{cases}$$

$$\delta_1 x = \begin{cases} x-3 & \text{if } x \equiv 1,2 \pmod 5 \\ x+3 & \text{if } x \equiv 3,4 \pmod 5 \\ x & \text{otherwise} \end{cases} \quad \delta_2 x = \begin{cases} x-3 & \text{if } x \equiv 4 \pmod 5 \\ x+3 & \text{if } x \equiv 1 \pmod 5 \\ x & \text{otherwise} \end{cases}$$

$$\begin{aligned} \delta_0\delta_2\mathcal{A}((9,8,7,5,3)) &= \{13,6,5,2,-1,-3,-4,-7,-8,\dots\} \\ &= \mathcal{A}(13,6,5,2) \\ \Rightarrow \delta_0\delta_2(9,8,7,5,3) &= (13,6,5,2) \end{aligned}$$



$(9, 8, 7, 5, 3)$ $\qquad$ $(13, 6, 5, 2) = \delta_0\delta_2(9, 8, 7, 5, 3)$

Proposition (Y)

The level q action of $a \in \mathfrak{W}_p$ on $\lambda \in \mathcal{B}$ has the following invariants:

1. $\overline{(a\lambda)}_q = \overline{\lambda}_q$
2. $\mathcal{Q}_p(a\lambda)$ *is the same as* $\mathcal{Q}_p(\lambda)$ *with the components reordered*
3. $\overline{wt}_p(a\lambda) = \overline{wt}_p(\lambda)$
4. $\overline{(a\lambda)}_p = a(\overline{\lambda}_p)$

Proposition (Olsson)

The q-bar-core of a p-bar-core is again a p-bar-core, or in the notation used earlier,

$$\overline{wt}_p(\lambda) = 0 \Rightarrow \overline{wt}_p(\overline{\lambda}_q) = 0$$

We generalise this result
(for the rest of this talk, p and q are coprime odd integers ≥ 3)

Proposition (Y)

For all $\lambda \in \mathcal{B}$,

$$\overline{wt}_p(\overline{\lambda}_q) \leq \overline{wt}_p(\lambda)$$

We are interested in the set $C_{p,q} := \{\lambda \in \mathcal{B} \mid \overline{\mathrm{wt}}_p(\lambda) = \overline{\mathrm{wt}}_p(\overline{\lambda}_q)\}$

Proposition (Y)

$$C_{p,q} = C_{q,p}$$

Proposition (Y)

$$a \in \mathfrak{W}_p,\ \lambda \in C_{p,q} \Rightarrow a\lambda \in C_{p,q}$$

Proof.

Using invariants (1) & (2) and the fact that $\lambda \in C_{p,q}$, we have

$$\overline{\mathrm{wt}}_p(\overline{(a\lambda)}_q) = \overline{\mathrm{wt}}_p(\overline{\lambda}_q) = \overline{\mathrm{wt}}_p(\lambda) = \overline{\mathrm{wt}}_p(a\lambda)$$

□

Interchanging p and q, we see that $C_{p,q} = C_{q,p}$ is also a union of orbits for the level p action of $\mathfrak{W}_q$

Hence $C_{p,q}$ is a union of orbits for the action of $\mathfrak{W}_p \times \mathfrak{W}_q$

The Υ-orbit

Proposition (Bessenrodt & Olsson)

There is a maximal bar partition $\Upsilon_{min\{p,q\},max\{p,q\}}$ in $C_p \cap C_q$, where $\Upsilon_{p,q}$ is the **Yin/Yang partition***, with parts*

$$(\tfrac{p-1}{2} - k)q - (l+1)p, \text{ for } k, l \in \mathbb{Z}_{\geq 0}$$

(maximal in the sense that $\lambda := \{\lambda_1, \lambda_2, \ldots\} \in C_p \cap C_q \Rightarrow \Upsilon_{\min\{p,q\},\max\{p,q\}} = \{\lambda_1 + a_1, \lambda_2 + a_2, \ldots\}$ with $a_1, a_2, \cdots \in \mathbb{Z}_{\geq 0}$)

$C^{\Upsilon}_{p,q} :=$ the orbit of $\Upsilon_{p,q}$ under the action of $\mathfrak{W}_p \times \mathfrak{W}_q$

Proposition (Y)

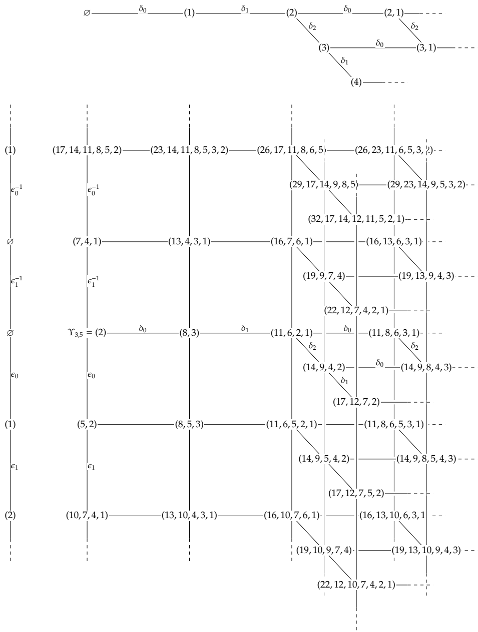
There is a bijection

$$C^{\Upsilon}_{p,q} \to 2^{\{1,\dots,(p-1)/2\}} \times C_p \times C_q$$

$$\lambda \mapsto (\{i \in \{1, \dots, \tfrac{p-1}{2}\} | \lambda_i^{(0 \ mod \ q)} q + p \in \mathcal{A}(\lambda)\}, \lambda^{(0 \ mod \ q)}, \lambda^{(0 \ mod \ p)})$$

Proposition (Y)

$\lambda \in \mathcal{B}, \ a \in \mathfrak{W}_p \Rightarrow (a\lambda)^{(0 \ mod \ q)} = a(\lambda^{(0 \ mod \ q)})$
where a acts at level q on $\mathcal{B}$, and at level 1 on C_p



The bijection between $\overline{C}_{3,5}^{\Upsilon}$ and $(\overline{C}_3 \times \overline{C}_3) \times \overline{C}_5$

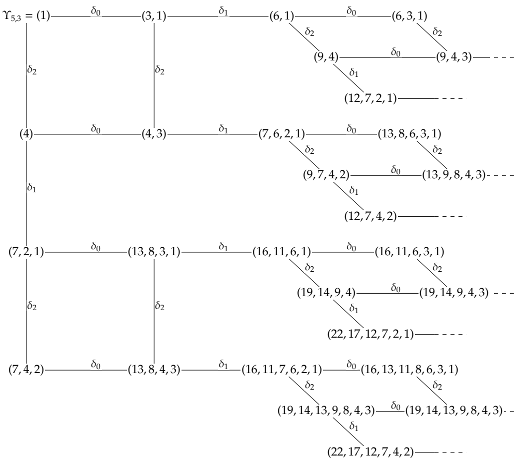


Figure 2: The branch of $\overline{C}^{\Upsilon}_{5,3}$ corresponding to the 3-bar-core $\varnothing$

Modular projective representation theory of S_n

We want to know the decomposition of $[\lambda]$ in a field of characteristic > 0.

Nazarov's construction gives us complex representating matrices for spin representations $[\lambda]$ (over 80 years after Schur's classification of irreducible characters!)

Using GAP, we can consider examples and try to identify decomposition factors, but this is difficult.

James described ordinary representations of S_n as submodules of induced modules to recover the Specht modules, a complete set of irreducible representations of S_n over $\mathbb{C}$.

Example

Contruct $(n-1,1)$ by starting with the natural module for S_n with basis $\{e_1,\dots,e_n\}$, where elements of $S_n = \langle s_1,\dots,s_{n-1}\rangle$ act by permuting the basis vectors:

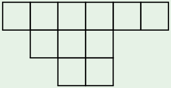
- $s_i e_i = e_{i+1}$,
- $s_i e_{i+1} = e_i$,
- $s_i e_j = e_j$ for $j \neq i, i+1$.

The Specht module $S^{(n-1,1)}$ is spanned by elements $e_i - e_{i+1}$ for $1 \leq i < n$. These are *polytabloids*.

Wales considers the induced module $[(n-1)]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n}$.

Morris' branching rule:
$[\lambda]$ is a composition factor of $[\mu]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n} \Leftrightarrow$ (the skew diagram of) λ is obtained by adding a node to (the diagram of) μ.
$[\lambda]$ is a composition factor of $[\nu]\downarrow_{\tilde{S}_n}^{\tilde{S}_{n+1}} \Leftrightarrow$ (the skew diagram of) λ is obtained by removing a node from (the diagram of) ν.

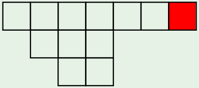
Example



Wales considers the induced module $[(n-1)]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n}$.

Morris' branching rule:
$[\lambda]$ is a composition factor of $[\mu]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n} \Leftrightarrow$ (the skew diagram of) λ is obtained by adding a node to (the diagram of) μ.
$[\lambda]$ is a composition factor of $[\nu]\downarrow_{\tilde{S}_n}^{\tilde{S}_{n+1}} \Leftrightarrow$ (the skew diagram of) λ is obtained by removing a node from (the diagram of) μ.

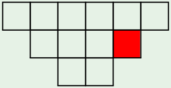
Example



Wales considers the induced module $[(n-1)]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n}$.

Morris' branching rule:
$[\lambda]$ is a composition factor of $[\mu]\uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n} \Leftrightarrow$ (the skew diagram of) λ is obtained by adding a node to (the diagram of) μ.
$[\lambda]$ is a composition factor of $[\nu]\downarrow_{\tilde{S}_n}^{\tilde{S}_{n+1}} \Leftrightarrow$ (the skew diagram of) λ is obtained by removing a node from (the diagram of) μ.

Example



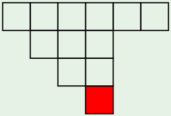
Wales considers the induced module $[(n-1)] \uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n}$.

Morris' branching rule:

$[\lambda]$ is a composition factor of $[\mu] \uparrow_{\tilde{S}_{n-1}}^{\tilde{S}_n} \Leftrightarrow$ (the skew diagram of) λ is obtained by adding a node to (the diagram of) μ.

$[\lambda]$ is a composition factor of $[\nu] \downarrow_{\tilde{S}_n}^{\tilde{S}_{n+1}} \Leftrightarrow$ (the skew diagram of) λ is obtained by removing a node from (the diagram of) μ.

Example



Now let M be a basic spin representation of $\tilde{S}_{n-1}$, so the generators t_i act via matrices T_i satisfying

$$T_i^2 = I, \quad T_i T_j = -T_j T_i \text{ for } |i-j| > 1, \quad T_i T_{i+1} + T_{i+1} T_i + I = 0.$$

Given $b \in M$, when we induce to $\tilde{S}_n$ we get an element

$$b_i = t_i t_{i+1} \cdots t_{n-1} b$$

in the induced module. For each i we'll regard $\{b_i | b \in M\}$ as a copy of M.

- $t_i b_i = b_{i+1}$;
- $t_i b_{i+1} = b_i$;
- $t_i b_j = (-1)^{n+j}(T_i b)_j$ if $j > i+1$;
- $t_i b_j = (-1)^{n+j}(T_{i-1} b)_j$ if $j < i$.

Now the copy of the module labelled by $(n-1, 1)$ inside here is spanned by elements of the form

$$b_i + (-1)^{n+i}(T_i b)_{i+1} + b_{i+2}$$

for $b \in M$ and $1 \leq i \leq n-2$.

We would like to do the same thing with the spin representation indexed by $(n-2,2)$. In the ordinary case, the Specht module is constructed as a submodule of the Young permutation module $M^{(n-2,2)}$, which can be thought of as the permutation module on 2-subsets of $\{1,\ldots,n\}$, or as the trivial module induced from the Young subgroup $S_{n-2}\times S_2$. If we take a basis $\{e_{ij}\}$, where $e_{ij}=e_{ji}$ is labelled by a pair $\{i,j\}$, then the Specht module is spanned by the polytabloids

$$e_{ij}-e_{ik}-e_{jl}+e_{kl}.$$

For the spin version, we induce the basic spin representation of the subgroup of $\tilde{S}_n$ generated by $t_1,\ldots,t_{n-3}$ and t_{n-1}. The induced module is spanned by elements

$$b_{ij}=t_jt_{j+1}\cdots t_{n-1}t_it_{i+1}\cdots t_{n-2}b$$

where $b\in M$ and $1\le i<j\le n$. (There are $\frac{n(n-1)}{2}$ distinct b_{ij} for each $b\in M$; by convention, $t_jt_{j+1}\cdots t_{n-1}=1$ when $j=n$, and $b_{n-1,n}:=b$.)