Verlinde rings

eigenfunctions

and

DAHA actions

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(BONN)

Rep $U_q(\mathfrak{g})$

Quantum groups at $\sqrt{1}$

H.H. Andersen

K_0

generalized
Θ-functions

Beauville

Kac-Moody
repr. of
fixed level

(Finkelberg, KL)

Verlinde rings

CFT
dim. of
conformal blocks

(Verlinde)

twisted K-theory

(Freedman-Hopkins-
Teleman)

3-mfd. invariants

(Reshetikhin-Turaev
Chern-Simons)

<u>Today</u>: Verlinde rings via
quantum groups!

Setup

$\mathfrak{g}$ complex simple Lie algebra or $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$C = (c_{ij})$ Cartan matrix

$$\boxed{D} := \max\{|c_{ij}| \quad i \neq j\} = \begin{cases} 1 & \text{types ADE} \\ 2 & \text{" BC} \\ 3 & \text{" G} \end{cases}$$

$\forall$ roots α $\quad d_\alpha := \frac{(\alpha,\alpha)}{2} \in \{1,2,3\}$

($d_\alpha = 1$ for simple roots)

$\rightsquigarrow \begin{pmatrix} d_{\alpha_1} & & \\ & \ddots & \\ & & d_{\alpha_r} \end{pmatrix}$ symmetrising matrix

$\rightsquigarrow$ Lusztig's divided power quantum group

$\quad U_q = U_q(\mathfrak{g})$

$$\boxed{q = \sqrt[\ell]{1}\,.}$$

ℓ could be even or odd

2 cases

Set $e' := \begin{cases} e & \text{if } e \text{ odd} \\ \frac{e}{2} & \text{if } e \text{ even} \end{cases}$

algebraists

topologists
physicists.

I) $D \nmid e'$

II) $\dot{D} \mid e'$

Rep U_q = finite dim repr. of U_q

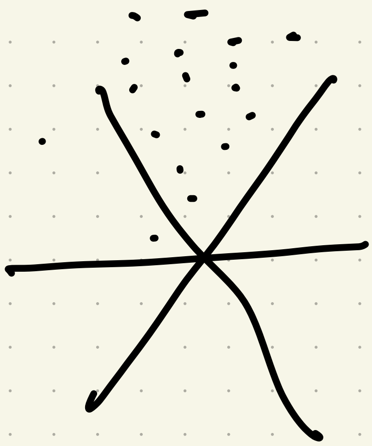
$\lambda \in X^+$ = dominant integral weights

$\rightsquigarrow \nabla(\lambda) := \mathrm{Ind}_{B_q}^{U_q} \lambda := F\left(\mathrm{Hom}_{B_q}(U_q, \lambda)\right)$

takes max. fd. submodule

$\Delta(\lambda) := \nabla(-w_0(\lambda))^*$

(dual) Weyl modules



Ex: $\mathfrak{g} = \mathfrak{sl}_2$ $\quad \Delta_q(3) = \Delta_q(3\varpi_1)$

1) q generic

$$\begin{tikzcd}
m_3 \arrow[loop above, "q^{-3}"] \arrow[r, bend left, "{[1]}"] & m_2 \arrow[loop above, "q^{-1}"] \arrow[l, bend left, "{[3]}"] \arrow[r, bend left, "{[1]}"] & m_1 \arrow[loop above, "q"] \arrow[l, bend left, "{[2]}"] \arrow[r, bend left, "{[3]}"] & m_0 \arrow[loop above, "q^3"] \arrow[l, bend left, "{[1]}"]
\end{tikzcd}$$

E

F

2) $q = \sqrt[3]{1}$, this is not irred.

$$L' \hookrightarrow \Delta(3\omega_1) \twoheadrightarrow L(3\omega_1)$$

$$L(3\omega_1) \hookrightarrow \nabla(3\omega_1) \twoheadrightarrow L'$$

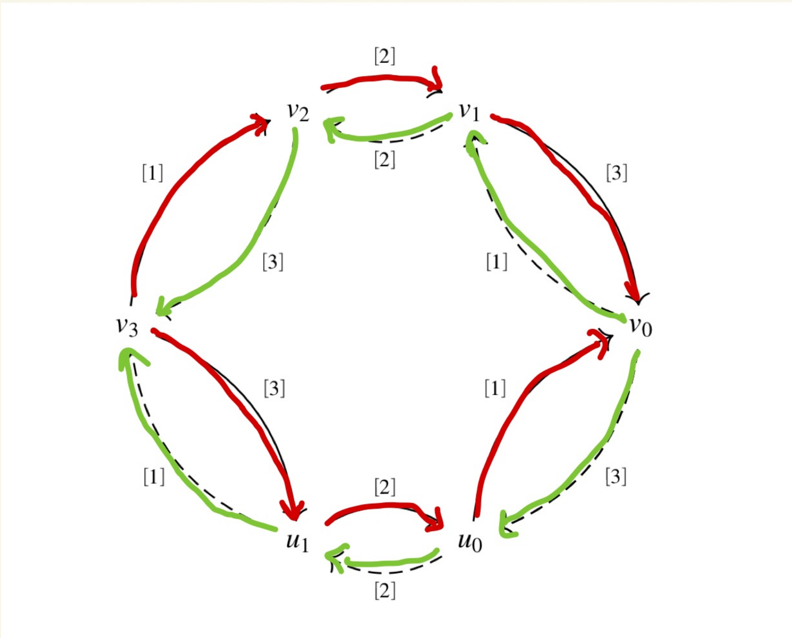
<u>Def</u>: $T \in \mathrm{Rep}\, U_q$ tilting

$\Leftrightarrow$ T has Δ-flag and has ∇-flag

<u>Ringel, Donkin, Andersen..</u> $T(\lambda)$, $\lambda \in X^+$
indec. tiltings

smaller
$\downarrow$
$\Delta(\nu)$s

$\Delta(\lambda)$



<u>Theorem</u> (Paradowski, Wenzl)

Tiltings form an additive monoidal ribbon category $\mathcal{T}$

Easy proof in type $\tilde{A}$ (Andersen - Tubbenhauer - S)

Claim: $\nabla(\lambda) \otimes \nabla(\mu)$ has ∇-flag $\forall \lambda, \mu$

$\Uparrow$

Claim' $\nabla(\lambda) \otimes \nabla(\omega_i)$ "

$\forall$ fund. weights ω_i

induction on partial order on weights

$\lambda = 0$ ✓

$\lambda \neq 0$ find $\eta = \lambda - \omega_i < \lambda$

$$\Pi \hookrightarrow \nabla(\eta) \otimes \nabla(\omega_i) \twoheadrightarrow \nabla(\lambda)$$

∇-flag with $\nabla(\nu)$ with $\nu < \lambda$

$$\Rightarrow \underbrace{\Pi \otimes \nabla(\mu)}_{\nabla\text{-flag by ind.}} \hookrightarrow \overbrace{\underbrace{\nabla(\eta) \otimes \nabla(\mu)}_{\nabla\text{-flag by ind.}} \otimes \nabla(\omega_i)}^{\nabla\text{-flag by Claim}'} \twoheadrightarrow \text{has } \nabla\text{-flag.}$$

Proof of Claim' uses:

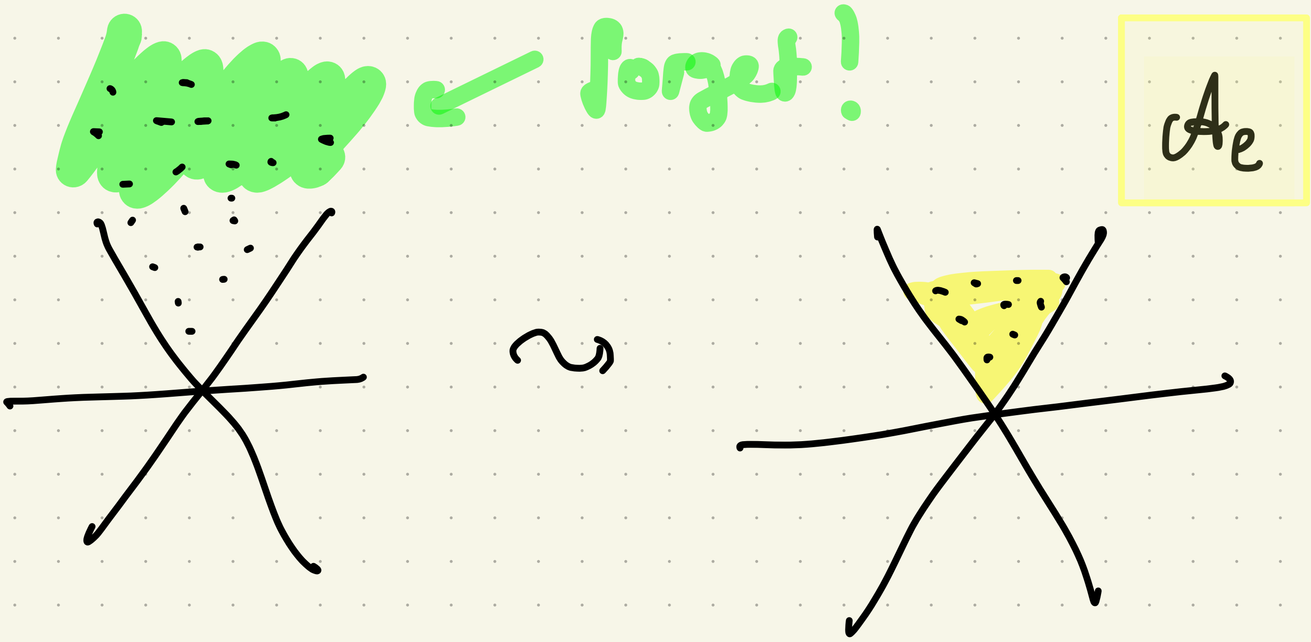
$$\nabla(\lambda) \otimes \nabla(\omega_i) \cong \mathrm{Ind}_{B_q}^{U_q} \left(\mathbb{C}_\lambda \otimes \nabla(\omega_i) \right)$$

minuscule!!

has ∇-flag

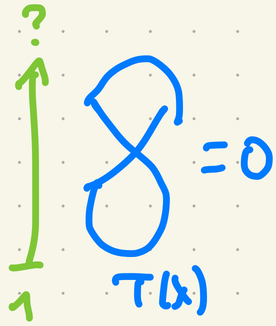
has B_q-module filtr with 1-dim subquots

Semisimplification



$\dot{T}(\lambda)$ neglidgible $\Leftrightarrow$ quantum dim $T(\lambda) = 0$

(i.e. $\mathrm{Tr}\,(K^{2\rho})\big|_{T(\lambda)} = 0$



)

$$\boxed{\mathcal{T}_{ss} := \mathcal{T}/_{\text{neglidgibles}}}$$

semisimple $\otimes$-cat.
(H.H. Andersen).

$$\mathcal{A}_e := \{\lambda \in X^+ \mid (\lambda+\rho, \theta_0) < e'\}$$

$$= \{\lambda \in X^+ \mid \langle \lambda+\rho, \theta_0^\vee \rangle < \frac{e'}{d_{\theta_0}}\}$$

$$\Theta_0 = \begin{cases} \Theta_{short} & \text{if } \mathrm{I}) \\ \Theta_{long} & \text{if } \mathrm{II}) \end{cases}$$

Coxeter number

Prop: (Sawin, Gajda-S.) Assume $\ell > h$

1) $\lambda \notin \mathcal{A}_\ell \Rightarrow T(\lambda)$ negligible

2) $[T(\lambda)]$ $\lambda \in \mathcal{A}_\ell$ form a basis of $K_0(\bar{\mathcal{T}}_{ss})$

finite except for gl_n

3) $\mathcal{A}_\ell$ fund. domain for affine Weyl group

$$W_\ell = \begin{cases} W \ltimes \ell' Q & \text{if } \mathrm{I}) \\ W \ltimes \ell' Q^\vee & \text{if } \mathrm{II}) \end{cases}$$

Jantzen

Physics, Kac, ..

What is $A := K_0(\mathcal{T}_{ss})$ as ring?

Andersen-S. : Presentation for ABCDG

<u>ex:</u> type C_n: $\mathbb{Z}[\chi(\omega_1), \ldots, \chi(\omega_n)] \Big/ \big(\chi(\ell\omega_1 + \omega_i)\big)_{1 \le i \le n}$

ℓ odd

Case $\mathfrak{g} = \mathfrak{gl}_n$

Theorem (Witten, Agnihotri, Korff-S.)

$$A \cong QH(Gr(n,e))$$

small quantum cohomology

$$[T(\lambda)] \longmapsto \text{Schubert class } \sigma_{\lambda^{(t)}}$$

$$\text{det repr.} \longleftarrow Q$$

Theorem (Verlinde, Siebert – Tian)

$$A \cong \mathbb{Z}[e_1, \dots, e_n, Q] / I \qquad e = k+n$$

elementary symmetric

$$I = (h_{k+1}, \dots, h_{e-1}, h_e + (-1)^n Q)$$

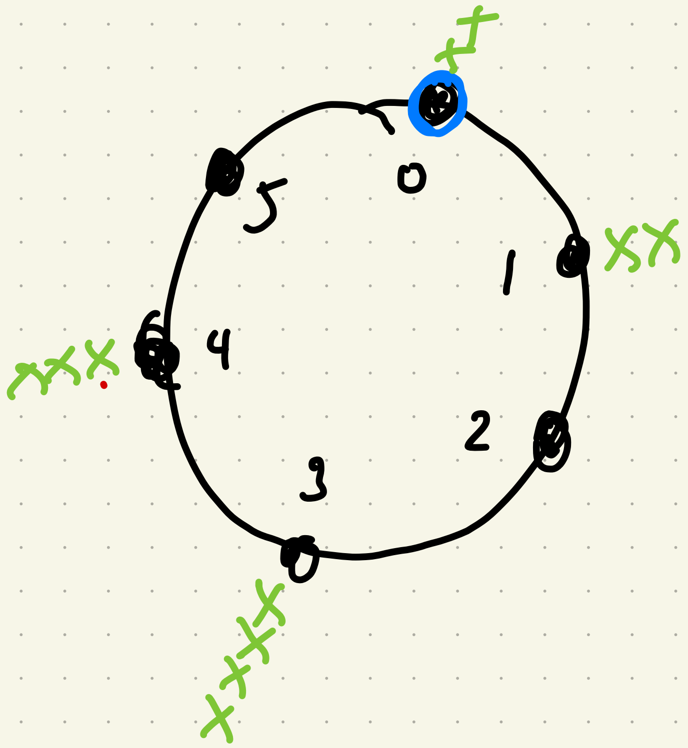
Observations:

- A quotient of character ring
- A "generically" semisimple

- Understanding the ring

$\widehat{=}$ understanding its spectrum

- encode highest weights or $[\Gamma(\lambda)]$ $\lambda \in A_\ell$ as "particle configurations" on (affine) Dynkin diagram with $\ell - n$ particles



$$\hat{\lambda} = 2\omega_1 + 0\omega_2 + 4\omega_3 + 3\omega_4 + 0\omega_5 + 2\omega_0$$

$$\text{level} = \ell - n$$

$$= 2 + 0 + 4 + 3 + 0 + 2$$

Operators $u_1, u_2, \dots, u_{n-1}, u_0$ of moving one particle to the next place (with factor Q for u_0)
from place i to $i+1$

Theorem (Korff – S.)

1) $\mathbb{Z}[Q]\left(\begin{array}{c}\text{particle conf.}\\ \text{of level } \ell\text{-}n\end{array}\right) \overset{\text{vsp}}{\cong} A = K_0(gl_n)$

at $\sqrt[\ell]{1}$

$$\lambda \longmapsto [\Delta(\lambda)]$$

2) The $u_1, \dots, u_{n-1}$ can be interpreted as crystal operators giving rise to the crystal of the $(\ell\text{-}n)$-th symmetric power of the vector repr. of $U_q(gl_n)$ generic

3) $\underbrace{S_\mu(u's)}_{\text{Schur functions in } u_i\text{'s}} . \lambda \longleftrightarrow [T(\mu) \otimes T(\lambda)]$

$\lambda, \mu \in A_\ell$

$S_\mu = \det(e_{\dots})$ elementary symm in u_i's

$e_r(u) = \sum_{|I|=r} \prod_{i \in I}^{\circlearrowleft} u_i$

Crucial: Construct a simultaneous eigenbasis for the action of symmetric polys in the u_i's.

over $\mathbb{Q}(\sqrt[e]{q})$

Tools: • Bethe Ansatz, integrable systems

• $\bigoplus_{i=0}^{n} A_{\text{level } i} \circlearrowleft$ "particle creation and annihilators"

module

Using these can construct eigenbasis.

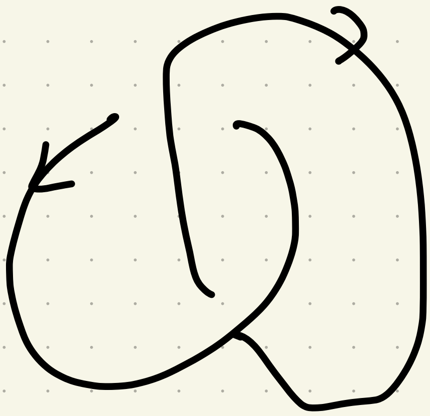
with eigenvalues as follows

$$e_r(u_i\text{'s}) \cdot b_\lambda = e_r[\ \ ]\, b_\lambda$$

↑ eigenbasis vector

normal symmetric poly in certain e-th roots of unities.

• Base change matrix: S-matrix



$T(\lambda)$ $T(\mu)$

endomorphism

$\mathbb{C}(q) \to \mathbb{C}(q)$ $\rightsquigarrow S = (s_{\lambda\mu})_{\lambda\mu}$

i.e. scalar $s_{\lambda\mu}$

$$\mathbb{1} \to T(\lambda) \otimes T(\lambda) \cdots \to \mathbb{1}$$

Tensor categories where S is invertible are modular

Modular tensor categories have $SL_2(\mathbb{Z})$-action on K_0

Cherednik's philosophy.

Any "Verlinde algebra" is a representation of a DAHA

Make this rigorous!

DAHA (Cherednik)

$\mathfrak{g}$ + any choice of lattice

$$\text{root lattice} \subset \Lambda \subset \text{weight lattice} \quad \rightsquigarrow \quad \text{DAHA } \mathbb{H}_{\tilde{q},t}$$

For us $L = X$.

Ex: $\mathfrak{g} = \mathfrak{gl}_2 \qquad \Lambda = \mathbb{Z} \oplus \mathbb{Z}i \subseteq \mathbb{C} \circlearrowleft \mathbb{Z}/2\mathbb{Z}$, $\; x \mapsto -x$

$E = \mathbb{C}/\Lambda \qquad E^* := E \setminus \{0\}$

$$\underbrace{\mathbb{C}\left[\pi_1\left(E^*/_{\mathbb{Z}/2\mathbb{Z}}\right)\right]}_{\text{group alg of DA braid group}} \xrightarrow[\text{relations}]{\text{quadr.}} \text{DAHA}$$

DAHA for $\mathfrak{gl}_n$ / GL_n

Def: $\mathbb{H}_{q,t}$ is the algebra generated by

$$T_0, T_1, \ldots, T_{n-1}, \pi^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1} \quad \text{modulo}$$

① $(T_i - t^{1/2})(T_i - t^{-1/2}) \quad \forall\ 0 \leq i \leq n-1$

② $T_i T_j = T_j T_i \qquad |i-j| > 1$

③ $T_i T_j T_i = T_j T_i T_j$

④ $\pi T_i \pi^{-1} = T_j$ for $j = i+1 \mod n$

⑤ $X_i X_j = X_j X_i \quad \forall\ 1 \leq i,j \leq n$

⑥ $T_i X_i T_i = X_{i+1}$

$T_0 X_n T_0 = q^{-1} X_1$

⑦ $T_i X_j = X_j T_i$ for $j \neq i, i+1 \mod n$

⑧ $\pi T_i \pi^{-1} = X_{i+1} \qquad 1 \leq i \leq n-1$

$\pi X_n \pi^{-1} = q^{-1} X_1$

(Irreducible) representations!?

Triangular decomposition: finite Hecke

$$\mathbb{H}_{\tilde{q},t} \overset{\text{vsp}}{\cong} k[X] \otimes \underbrace{\mathcal{H}_t \otimes k[X^\vee]}_{= B \text{ subalgebra}}$$

$$\boxed{\mathrm{Pol}} := \mathbb{H}_{\tilde{q},t} \otimes_B \mathrm{triv} \cong k[X] \cong k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

for gl_n

Verma module / Polynomial representation

$$\boxed{\text{Specialise:} \quad \tilde{q} = t = e^{\frac{2\pi i}{e}}}$$

$\mathbb{H} := \mathbb{H}_{\tilde{q},t}$ for these values.

Connection to representation ring

Let e = symmetrizing idempotent in $\mathcal{H}_t \subseteq \mathbb{H}$

$e\mathbb{H}e$ spherical DAHA $\curvearrowright$ $e\,\mathrm{Pol}$

__Theorem__ (Gajda – S.) For all types

1) $e\,\mathrm{Pol} \cong \mathbb{k}[X]^W$ ← Weyl group

2) $\mathcal{M} := e\,\mathrm{Pol} / e\,\mathrm{Red}$ gives

an irreducible repr of $e\mathbb{H}e$

__Theorem__ (Gajda – S.) In type A:

1) $\mathcal{M}$ decomposes into 1-dim

eigenspaces for the action of

$\mathbb{k}[X^\vee]$

2) For $\mathfrak{g}=\mathfrak{gl}_n$

$$\begin{tikzcd}
 & & K_0(\mathcal{J}_{ss}) \arrow[d, phantom, "\cong" rotate=90] \\
I \arrow[r, hook] \arrow[d, dashed, "?"] & \mathbb{C}[e_1,\dots,e_n,Q] \arrow[r, two heads] \arrow[d] & QH(Gr(k,n)) \arrow[d, "\int"] \\
eRad \arrow[r, hook] & \mathbb{C}[X]^W = \mathbb{C}[X_1^{\pm 1},\dots,X_n^{\pm 1}]^{S_n} \arrow[r, two heads] & \mathcal{M}
\end{tikzcd}$$

With $\tilde{q} = q^{1/2}$

In general:

Only generalised eigen space decomp

and $K_0(\mathcal{J}_{ss}) \twoheadrightarrow \mathcal{M}$ "Correct" semisimplification

ePol

using results of Brugières. one can realise $\mathcal{M}$ as K_0 of some (possibly spin) modular tensor cat.

$\Rightarrow$ SDAHA detects modularity

Savin: The category $\mathcal{T}_{ss}$ is modular except in the following cases:

	A_n	B_n	C_n	D_n	E_7
ℓ even	✓	if $D \neq e'$, $2 \nmid n$ ω_n	D / e' ω_1	✓	✓
ℓ odd	if $2 \nmid n$ $\omega_{\frac{n+1}{2}}$	if $2 \mid n$ ω_n	always	always ω_1 if $2 \nmid n$ ω_1, ω_{n-1} if $2 \mid n$	always ω_7

weights giving rise to generalised eigenspaces.

Thanks for Listening!

- Korff - S. : The $\mathfrak{sl}_n$ - WZNW fusion ring
  A combinatorial construction and a realisation as quantum cohomology
  (Advances)
- Andersen - S. : Fusion rings for quantum groups
  (Repr. Theory)
- Gajda - S. : DAHA and their representations and fusion ring in type A
  (Master Thesis)