Groups of p-central type

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Normal subgroups

- Let G be a finite group with a normal subgroup N.
- Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G.
- For $\chi \in \mathrm{Irr}(G)$ let χ_N be the restriction of χ to N.

Theorem (Clifford)

For every $\chi \in \mathrm{Irr}(G)$ there exist $e \in \mathbb{N}$ and $\theta_1, \ldots, \theta_k \in \mathrm{Irr}(N)$ such that

$$\chi_N = e(\theta_1 + \ldots + \theta_k).$$

Moreover, $\{\theta_1, \ldots, \theta_k\}$ is a G-orbit under conjugation.

Clifford correspondence

- We call e the ramification index of χ with respect to N.
- For $\theta \in \mathrm{Irr}(N)$ let G_θ be the stabilizer of θ in G.
- Let $\mathrm{Irr}(G|\theta)$ be the set of $\chi \in \mathrm{Irr}(G)$ such that θ is a constituent of χ_N.

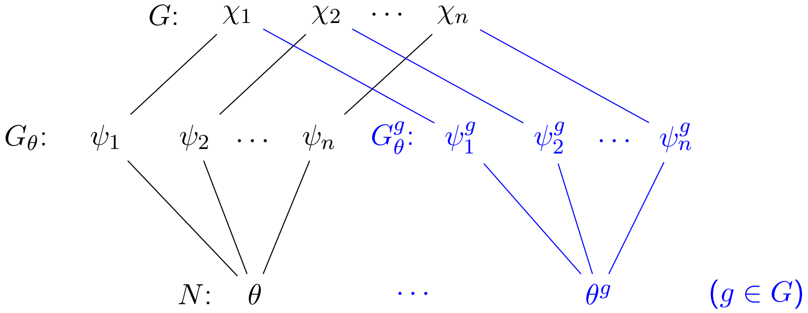
Theorem (Clifford correspondence)

For $\theta \in \mathrm{Irr}(N)$ the map

$$\mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(G|\theta), \qquad \psi \mapsto \psi^G$$

is a bijection, which preserves the ramification index.

Diagram



G: χ_1 χ_2 $\cdots$ χ_n
G_θ: ψ_1 ψ_2 $\cdots$ ψ_n
G_θ^g: ψ_1^g ψ_2^g $\cdots$ ψ_n^g
N: θ $\cdots$ θ^g
$(g \in G)$

Extensions

- Observation: $e = 1$ if and only if θ extends to $\mathrm{Irr}(G_\theta)$.
- Then $\mathrm{Irr}(G|\theta)$ is completely determined by:

Theorem (Gallagher)

If $\hat{\theta} \in \mathrm{Irr}(G_\theta)$ is an extension of $\theta \in \mathrm{Irr}(N)$, then

$$\mathrm{Irr}(G_\theta|\theta) = \left\{\lambda\hat{\theta} : \lambda \in \mathrm{Irr}(G_\theta/N)\right\}.$$

Extensions

Example

In each of the following situations $\theta \in \mathrm{Irr}(N)$ extends to G_θ:

- $\gcd(|N|, |G_\theta : N|) = 1$.
- all Sylow subgroups of G_θ/N are cyclic.
- $\mathrm{H}^2(G_\theta/N, \mathbb{C}^\times) = 0$ (Schur multiplier).

Bounds on the ramification index

- Now we consider the opposite situation, where $e = e_\chi$ is large.
- By Frobenius reciprocity,

$$\theta^{G_\theta} = \sum_{\chi\in\mathrm{Irr}(G_\theta|\theta)} e_\chi \chi,$$

$$|G_\theta : N|\theta(1) = \theta^{G_\theta}(1) = \sum_{\chi\in\mathrm{Irr}(G_\theta|\theta)} e_\chi \chi_N(1) = \theta(1) \sum_{\chi\in\mathrm{Irr}(G_\theta|\theta)} e_\chi^2,$$

$$|G_\theta : N| = \sum_{\chi\in\mathrm{Irr}(G_\theta|\theta)} e_\chi^2.$$

Fully ramified characters

Theorem

For every $\chi \in \mathrm{Irr}(G)$ the following assertions are equivalent:

1. $\chi_N = e\theta$ *for some* $\theta \in \mathrm{Irr}(N)$ *and* $e^2 = |G : N|$.
2. $e\chi = \theta^G$ *for some* $\theta \in \mathrm{Irr}(N)$ *and* $e \in \mathbb{N}$ *with* $G_\theta = G$.

Definition

In the situation of the theorem, we call χ and θ fully ramified.

Fully ramified characters

Example

Let

$$G = \langle x, y \mid x^4 = y^2 = 1,\ x^y = x^{-1} \rangle \cong D_8,$$
$$N = \mathrm{Z}(G) = \langle x^2 \rangle \cong C_2$$

and $1 \neq \theta \in \mathrm{Irr}(N)$. Then θ is fully ramified and $\theta^G = 2\chi$.

G	1	x^2	x	y	xy
λ_1	1	1	1	1	1
λ_2	1	1	1	-1	-1
λ_3	1	1	-1	1	-1
λ_4	1	1	-1	-1	1
χ	2	-2	0	0	0

Groups of central type

- Using character triples, we can often assume that $N = \mathrm{Z}(G)$.
- Then $\chi(1)^2 = e_\chi^2\theta(1)^2 = e_\chi^2 \leq |G : \mathrm{Z}(G)|$ for every $\chi \in \mathrm{Irr}(G)$.

Definition

A group G is of central type if the following equivalent conditions hold:

1. There exists some $\chi \in \mathrm{Irr}(G)$ with $\chi(1)^2 = |G : \mathrm{Z}(G)|$.
2. There exists a fully ramified character $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$.

Groups of central type

Theorem (Howlett–Isaacs, 1982)

Groups of central type are solvable.

- The theorem was conjectured in 1964 by Iwahori and Matsumoto.
- There was an incomplete proof attempt by Liebler and Yellen, 1979.
- The proof depends on the classification of finite simple groups.

Theorem (Gagola)

For every solvable group H there exists a group G of central type such that H is a subgroup of $G/\mathrm{Z}(G)$.

Groups of central type

The following is still open:

Conjecture (Humphreys)

If there exists $\theta \in \mathrm{Irr}(\mathrm{Z}(G))$ such that all irreducible constituents of θ^G have the same degree, then G is solvable.

- This was proved by Higgs if θ^G has (at most) two constituents.
- The conjecture also holds if the constituents of θ^G are conjugate in $\mathrm{Aut}(G)$ by Navarro–Rizo.

Brauer characters

- Replace $\mathbb{C}$ by an algebraically closed field F of characteristic $p > 0$.
- Let $G_{p'}$ be the set of p'-elements of G (order coprime to p).
- The F-representations of G correspond to Brauer characters $\chi\colon G_{p'} \to \mathbb{C}$.
- Let $\mathrm{IBr}(G)$ be the set of irreducible Brauer characters of G.
- If $p \nmid |G|$, then $G_{p'} = G$ and $\mathrm{Irr}(G) = \mathrm{IBr}(G)$.
- Clifford theory works in the same way for Brauer characters.

Groups of p-central type

Definition

We call G of p-central type if there exists a fully ramified Brauer character $\theta \in \mathrm{IBr}(\mathrm{Z}(G))$, i. e. $\theta^G = e\chi$ for some $\chi \in \mathrm{IBr}(G)$ and $e \in \mathbb{N}$.

Theorem (S., 2023)

Groups of p-central type are solvable.

- The theorem was proved by Navarro–Späth–Tiep for $p \neq 5$.
- For $p = 5$, they show that a minimal counterexample has order $> 10^{46}$ and involves $6.A_6$.
- The theorem implies Howlett–Isaacs by choosing $p \nmid |G|$.

Groups of p-central type

Proposition (Navarro–Späth–Tiep)

For every solvable group G the following assertions are equivalent:

1. *G has p-central type.*
2. *There exists $\chi \in \mathrm{IBr}(G)$ such that $\chi(1)^2 = |G : \mathrm{Z}(G)|_{p'}$.*

This is false for the non-solvable group $G = \mathrm{SL}(2, 17)$ with $p = 17$.

Theorem (S.)

For every solvable group H and every prime p there exists a group G of p-central type such that H is a subgroup of $G/\mathrm{Z}(G)$.

Blocks

- The group algebra FG splits into indecomposable ideals

$$FG = B_1 \oplus \ldots \oplus B_n,$$

which are called the p-blocks of G.

- Each block is an F-algebra and the irreducible representations of G distribute into blocks

$$\operatorname{Irr}(G) = \operatorname{Irr}(B_1) \,\dot{\cup}\, \ldots \,\dot{\cup}\, \operatorname{Irr}(B_n),$$
$$\operatorname{IBr}(G) = \operatorname{IBr}(B_1) \,\dot{\cup}\, \ldots \,\dot{\cup}\, \operatorname{IBr}(B_n).$$

- If $p \nmid |G|$, then $|\operatorname{Irr}(B_i)| = |\operatorname{IBr}(B_i)| = 1$ for $i = 1, \ldots, n$.

Blocks of solvable groups

Let $N := \mathrm{O}_{p'}(G)$ be the largest normal p'-subgroup of G.

Theorem (Fong)

Let B be a block of a solvable group G with $\chi \in \mathrm{Irr}(B)$. Let $\theta \in \mathrm{Irr}(N) = \mathrm{IBr}(N)$ be a constituent of χ_N. Then $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$ and $\mathrm{IBr}(B) = \mathrm{IBr}(G|\theta)$.

Corollary

If θ is fully ramified as a Brauer character, then $l(B) := |\mathrm{IBr}(B)| = 1$.

Question (Kessar–Linckelmann)

Is every block B with $l(B) = 1$ Morita equivalent to a block of a solvable group?

Nilpotent blocks

- If G/N is a p-group, then every $\theta \in \mathrm{IBr}(N)$ is fully ramified by Green's theorem.
- Blocks arising in this way are called nilpotent.
- Every nilpotent block is Morita equivalent to the group algebra of a p-group by Puig's theorem.

Example

Let $G = S_3$ and let θ be the trivial character of $N = \mathrm{O}_{2'}(G) = A_3$. Then the principal 2-block B of G is nilpotent, $B \cong F[G/N] \cong FC_2$ and

$$\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta) = \mathrm{Irr}(G/N) = \{1_G, \mathrm{sgn}\}.$$

Non-nilpotent blocks

- So far, all non-nilpotent blocks with $l(B) = 1$ were constructed for solvable groups G with abelian Sylow p-subgroups.
- In this case $G/\mathrm{O}_{p'}(G)$ has a normal Sylow p-subgroup (Hall–Higman lemma).
- In general, let $\mathrm{O}_{pp'}(G)/\mathrm{O}_{p'}(G) := \mathrm{O}_p(G/\mathrm{O}_{p'}(G))$.

Definition

For a solvable group G, define the p-length $l_p(G)$ inductively:

$$l_p(G) := \begin{cases} 0 & \text{if } p \nmid |G|, \\ 1 + l_p(G/\mathrm{O}_{pp'}(G)) & \text{otherwise.} \end{cases}$$

Non-nilpotent blocks

Theorem (S.)

There exist solvable groups with arbitrarily large p-length and non-nilpotent p-blocks B (of maximal defect) with $l(B) = 1$.

Proof.

- Let H be of p-central type with "large" p-length and $Z := \mathrm{Z}(H)$.
- Let V be an $\mathbb{F}_p$-vector space on which H/Z acts faithfully.
- Then $G = V \rtimes H$ has a non-nilpotent block B (of maximal defect) with $l(B) = 1$. □

Non-nilpotent blocks

Example

The group

$$H := \texttt{SmallGroup}(54, 8) \cong 3_+^{1+2} \rtimes C_2$$

is of 2-central type with $Z \cong C_3$ and H/Z acts faithfully on $V \cong \mathbb{F}_2^4$. Now

$$G := V \rtimes H = \texttt{SmallGroup}(864, 3996)$$

has 2-length 2 and possess a non-nilpotent 2-block B with defect 5 and $l(B) = 1$.

Lifting Brauer characters

- For $\chi \in \mathrm{Irr}(G)$ the restriction χ^0 to $G_{p'}$ is a Brauer character of G.
- Conversely, every $\varphi \in \mathrm{IBr}(G)$ is an integral linear combination of characters χ^0 for some $\chi \in \mathrm{Irr}(G)$.

Conjecture (basic sets)

For every block B of G there exists $C \subseteq \mathrm{Irr}(B)$ such that $\{\chi^0 : \chi \in C\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\mathrm{IBr}(B)$.

For solvable groups, the conjecture follows from the Fong–Swan theorem: every $\varphi \in \mathrm{IBr}(G)$ lifts to $\mathrm{Irr}(G)$, i. e. $\varphi = \chi^0$ for some $\chi \in \mathrm{Irr}(G)$.

Lifting Brauer characters

This does not hold for non-solvable groups:

Example

For the principle 2-block B of A_5 we have $\mathrm{Irr}(B) = \{\chi_1, \dots, \chi_4\}$ and $\mathrm{IBr}(B) = \{\chi_1^0,\ \chi_2^0 - \chi_1^2,\ \chi_3^0 - \chi_1^0\}$.

	1	x_2	x_3	x_5	x_5^2
χ_1	1	1	1	1	1
χ_2	3	-1	0	ω	ω^*
χ_3	3	-1	0	ω^*	ω
χ_4	5	1	-1	0	0

	1	x_3	x_5	x_5^2
φ_1	1	1	1	1
φ_2	2	-1	$-\omega^*$	$-\omega$
φ_3	2	-1	$-\omega$	$-\omega^*$

where $\omega := \frac{1+\sqrt{5}}{2}$ and $\omega^* := \frac{1-\sqrt{5}}{2}$.

Unique lifts

Kessar–Linckelmann's question motivates the following theorem:

Theorem (Malle–Navarro–Späth)

If $l(B) = 1$ *then* $\mathrm{IBr}(B) = \{\chi^0\}$ *for some* $\chi \in \mathrm{Irr}(B)$.

- The proof relies (again) on the classification of finite simple groups.
- If B is nilpotent, then $\varphi \in \mathrm{IBr}(B)$ has at least p lifts unless $|\mathrm{Irr}(B)| = 1$.
- Navarro has asked me whether there exists blocks B with $l(B) = 1$ such that $\varphi \in \mathrm{IBr}(B)$ has a unique lift.
- If a group H acts on a vector space V, we denote the stabilizer of $v \in V$ in H by H_v.

Unique lifts

Theorem (S.)

Let H be a p'-group of central type such that $H/\mathrm{Z}(H)$ acts faithfully on an $\mathbb{F}_p$-vector space V. Suppose $|H : H_v| > |H_v : \mathrm{Z}(H)|$ for every $v \in V \setminus \{0\}$. Then $G := V \rtimes H$ has a p-block B with $\mathrm{IBr}(B) = \{\varphi\}$ such that φ has a unique lift.

Example

The group

$$H := \texttt{SmallGroup}(128, 144)$$

has central type and $H/\mathrm{Z}(H)$ or order 64 acts faithfully on $V = \mathbb{F}_5^4$ with the desired property. This yields a 5-block of G with $|G| = 40{,}000$.