Content systems and KLR algebras

Andrew Mathas

OIST, August 2022

Joint work with Anton Evseev

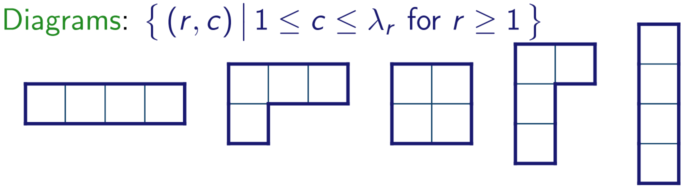
Symmetric group combinatorics

Let $\mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$

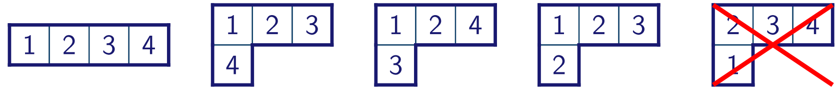
$\Longrightarrow$ the conjugacy classes of $\mathfrak{S}_n$ are indexed by the *partitions* of n

Example The partitions when $n = 4$ are $\{(4), (3,1), (2^2), (2,1^2), (1^4)\}$

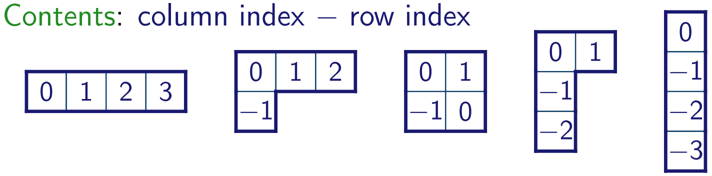
- Diagrams: $\{\,(r,c) \mid 1 \le c \le \lambda_r \text{ for } r \ge 1\,\}$



- Standard λ-tableaux $\mathrm{Std}(\lambda)$: increase along rows and down columns



- Contents: column index − row index



Young's seminormal form

Young's seminormal form (1901)

Let λ be a partition of n and let V_λ be the $\mathbb{Q}$-vector space with basis $\{\, v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\,\}$ and $\mathfrak{S}_n$-action

$$(r, r+1)v_{\mathfrak{t}} = \frac{1}{c_r(\mathfrak{t}) - c_{r+1}(\mathfrak{t})} v_{\mathfrak{t}} + \frac{c_r(\mathfrak{t}) - c_{r+1}(\mathfrak{t}) + 1}{c_r(\mathfrak{t}) - c_{r+1}(\mathfrak{t})} v_{\mathfrak{s}},$$

where $\mathfrak{s} = (r, r+1)\mathfrak{t}$ and $c_k(\mathfrak{u})$ is the content of k in $\mathfrak{u}$

Then V_λ is an absolutely irreducible $\mathbb{Q}\mathfrak{S}_n$-module

Key points

- Contents separate tableaux: $\mathfrak{s} = \mathfrak{t} \iff c_k(\mathfrak{s}) = c_k(\mathfrak{t})$ for $1 \le k \le n$
- The $v_{\mathfrak{t}}$ are eigenvectors for the Jucys-Murphy elements:

$$L_k v_{\mathfrak{t}} = c_k(\mathfrak{t}) v_{\mathfrak{t}}, \text{ where } L_k = \sum_{j=1}^{k-1} (j,k) \in \mathbb{Q}\mathfrak{S}_n$$

- $F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{\mathfrak{s} \text{ standard} \\ c_k(\mathfrak{t}) \ne c_k(\mathfrak{s})}} \frac{L_k - c_k(\mathfrak{s})}{c_k(\mathfrak{t}) - c_k(\mathfrak{s})}$ is a primitive idempotent in $\mathbb{Q}\mathfrak{S}_n$

James' Specht modules

In 1976, Gordon James constructed the Specht module S_λ of the symmetric group over a field F as a submodule of $M_\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(1)$

$\Longrightarrow$ S_λ comes equipped with an associative inner product $\langle\ ,\ \rangle$

$\Longrightarrow$ $\mathrm{rad}\, S_\lambda = \{\, a \in S_\lambda \mid \langle a, b\rangle = 0 \text{ for all } b \in S_\lambda\,\}$ is a submodule

Theorem (James 1976)

The module $D_\lambda = S_\lambda / \mathrm{rad}\, S_\lambda$ is either zero or absolutely irreducible

Key points

- First construction of the irreducible $\mathfrak{S}_n$-modules over an arbitrary field
- The inner product on S_λ is crucial
- Recovers $D_\lambda = S_\lambda$ in characteristic zero
- No content functions in sight!
- The Specht module S_λ has a basis indexed by standard tableaux
- There is an implicit interplay between $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(1)$ and $\mathrm{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n}(\mathrm{sgn})$

Seminormal forms for KLR algebras

In 2008, Brundan and Kleshchev (and Rouquier), proved that $F\mathfrak{S}_n$ is isomorphic to a cyclotomic KLR algebra $\mathscr{R}_n^{\Lambda_0} = \langle 1_{\mathrm{i}}, \psi_r, y_k \,|\, \dots \rangle$

Proposition (Brundan and Kleshchev 2008)

If char $F = p > n$ then there is a graded irreducible $\mathscr{R}_n^{\Lambda_0}$-module with basis $\{ v_{\mathfrak{t}} \,|\, \mathfrak{t} \in \mathrm{Std}(\lambda) \}$ and $\mathscr{R}_n^{\Lambda_0}$-action

$$1_{\mathrm{i}} v_{\mathfrak{t}} = \delta_{\mathrm{i},\mathrm{c}(\mathfrak{t})} v_{\mathfrak{t}}, \quad y_k v_{\mathfrak{t}} = 0 \quad \text{and} \quad \psi_r v_{\mathfrak{t}} = v_{\mathfrak{s}}$$

where $\mathfrak{s} = (r, r+1)\mathfrak{t}$ and $\mathrm{c}(\mathfrak{t}) = \big(\mathrm{c}_1(\mathfrak{t}), \dots, \mathrm{c}_n(\mathfrak{t})\big)$

Key points

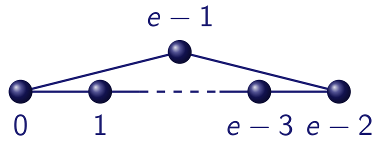
- Using the Brundan-Kleshchev graded isomorphism theorem, this is a direct translation of Young's seminormal form: $1_{\mathrm{c}(\mathfrak{t})} = F_{\mathfrak{t}}$
- (2011) Brundan-Kleshchev-Wang introduced graded Specht modules
- No inner product in sight
- (2010) Hu-M. showed that $\mathscr{R}_n^{\Lambda}$ is a graded cellular algebra in type A
- In general, replace contents by residues $\mathrm{r}(\mathfrak{t}) := \mathrm{c}(\mathfrak{t}) \pmod{p}$.

Symmetrisable Quivers

Let Γ be a symmetrisable quiver

We are mainly interested in the following quivers:

$A^{(1)}_{e-1}$, $e \geq 2$ $\qquad$ $C^{(1)}_{e-1}$, $e \geq 3$



Both quivers have vertex set $I = \{0, 1, \dots, e-1\}$

To the quiver Γ we attach:

- Fundamental weights $\{\Lambda_i\}$, simple roots $\{\alpha_i\}$, simple coroots $\{\alpha_i^\vee\}$
- A Cartan matrix $C = (c_{ij})_{i,j \in I}$ and bilinear form $(\alpha_i^\vee, \alpha_j) = d_i c_{ij}$
- Positive and dominant root lattices: $Q^+ = \bigoplus_i \mathbb{N}\alpha_i$ and $P^+ = \bigoplus_i \mathbb{N}\Lambda_i$
- A quantised Kac-Moody algebra $U_q(\Gamma)$

Rouquier's Q-polynomials

Let $K = \bigoplus_{d \in \mathbb{Z}} K_d$ be a graded commutative ring with indeterminates u, v

Following Rouquier, fix a family of Q-polynomials $Q_{ij}(u,v) \in K[u,v]$ such that $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, $Q_{i,i}(u,v) = 0$ and if $i \neq j$ then

$$Q_{i,j}(u,v) = \sum_{p,q \geq 0} t_{i,j;p,q} u^p v^q, \quad \text{where } t_{i,j,-c_{ij},0} \in K_0^\times$$

and $t_{i,j;p,q} \in K_d$ for $d = -2\langle \alpha_i, \alpha_j \rangle - p\langle \alpha_i, \alpha_i \rangle - q\langle \alpha_j, \alpha_j \rangle$.

Example The standard choice for these polynomials is $K = \Bbbk$ and

$$Q_{i,j}(u,v) = \begin{cases} u - v & \text{if } i \to j, \\ (u-v)(v-u) & \text{if } i \leftrightarrows j, \\ u - v^2 & \text{if } i \Rightarrow j. \end{cases}$$

We will also use $K = \Bbbk[x]$, or $\Bbbk[x^{\pm}] = \Bbbk[x, x^{-1}]$, and polynomials like:

$$Q^x_{i,j}(u,v) = \begin{cases} u - v - x^2 & \text{if } i \to j, \\ (u - v + x^2)(v - u + x^2) & \text{if } i \leftrightarrows j, \\ u - (v - x^2)^2 & \text{if } i \Rightarrow j. \end{cases}$$

Cyclotomic Khovanov-Lauda-Rouquier algebras

Fix $\alpha \in Q^+$, $\Lambda \in P^+$ and $n \geq 0$. Let $I^\alpha = \{ \mathrm{i} \in I^n \,|\, \alpha = \sum_k \alpha_{i_k} \}$

The cyclotomic KLR algebra $\mathscr{R}_\alpha^\Lambda$ is the unital associative K-algebra generated by $\{ 1_{\mathrm{i}} \,|\, \mathrm{i} \in I^\alpha \} \cup \{ \psi_k \,|\, 1 \leq k < n \} \cup \{ y_k \,|\, 1 \leq k \leq n \}$ subject to the relations:

- $1_{\mathrm{i}} 1_{\mathrm{j}} = \delta_{\mathrm{i},\mathrm{j}} 1_{\mathrm{i}}$, $\quad y_k 1_{\mathrm{i}} = 1_{\mathrm{i}} y_k$, $\quad y_k y_m = y_m y_k$, $\quad \sum_{\mathrm{i} \in I^\alpha} 1_{\mathrm{i}} = 1$
- $\psi_k 1_{\mathrm{i}} = 1_{r_k \mathrm{i}} \psi_k$, $\quad \psi_k \psi_m = \psi_m \psi_k$ if $|m - k| > 1$
- $(\psi_k y_{k+1} - y_k \psi_k) 1_{\mathrm{i}} = \delta_{i_k, i_{k+1}} 1_{\mathrm{i}} = (y_{k+1} \psi_k - \psi_k y_k) 1_{\mathrm{i}}$
- $\psi_k^2 1_{\mathrm{i}} = Q_{i_k, i_{k+1}}(y_k, y_{k+1}) 1_{\mathrm{i}}$
- $(\psi_{k+1} \psi_k \psi_{k+1} - \psi_k \psi_{k+1} \psi_k) 1_{\mathrm{i}} = \delta_{i_k, i_{k+2}} \frac{Q_{i_k, i_{k+1}}(y_k, y_{k+1}) - Q_{i_{k+1}, i_k}(y_{k+1}, y_{k+2})}{y_k - y_{k+2}}$
- $W_{i_1}(y_1) 1_{\mathrm{i}} = 0$, for $\mathrm{i} \in I^\alpha$ and suitable weight polynomials $W_i(u)$

Set $\mathscr{R}_n^\Lambda = \bigoplus_\alpha \mathscr{R}_\alpha^\Lambda$, where $\alpha \in Q^+$ has height n

Importantly, $\mathscr{R}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra with degree function

$$\deg 1_{\mathrm{i}} = 0, \quad \deg y_m 1_{\mathrm{i}} = (\alpha_{i_m}, \alpha_{i_m}) \text{ and } \deg \psi_k 1_{\mathrm{i}} = -(\alpha_{i_k}, \alpha_{i_{k+1}})$$

Type $A^{(1)}_{e-1}$ (Brundan-Kleshchev): $\mathscr{R}_n^\Lambda$ is a cyclotomic Hecke algebra

Content systems for KLR algebras

Let Γ_ℓ be the quiver of type $A_\infty^{\times\ell}$ with vertices $J_\ell = \{1,\dots,\ell\} \times \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}^\ell$

Let $\Bbbk$ be a commutative ring, $\mathbb{K}$ its field of fractions and consider the graded rings $\Bbbk[x]$ and $\mathbb{K}[x^\pm] = \mathbb{K}[x, x^{-1}]$

A content system for (Q, W) with values in $\Bbbk[x]$ is a pair of functions

$$\mathrm{c}: J_\ell \longrightarrow \Bbbk[x] \quad \text{and} \quad \mathrm{r}: J_\ell \longrightarrow I$$

such that

1. $W_i(u) = \prod_{\mathrm{r}(l,0)=i} (u - \mathrm{c}(l,0))$
2. If $i = \mathrm{r}(k,a)$ and $j \in \{\mathrm{r}(k, a \pm 1)\}$ then, for some $\varepsilon = \varepsilon_{ij} \in \Bbbk^\times$,
$$Q_{i,j}(\mathrm{c}(k,a), v) = \varepsilon \prod_{b=a\pm1, \mathrm{r}(k,b)=j} (\mathrm{c}(k,b) - v)$$
3. If $(k,a), (l,b) \in J_\ell$ with $-n < a, b < n$ then $(k,a) = (l,b)$ if and only if $\mathrm{c}(k,a) = \mathrm{c}(l,b)$ and $\mathrm{r}(k,a) = \mathrm{r}(l,b)$

Key points

- The content system encodes the polynomials defining $\mathrm{R}_n^\Lambda = \mathrm{R}_n^\Lambda(Q, W)$
- In Young's seminormal form, content sequences separate tableaux
  - $\rightsquigarrow$ Now content and residue sequences separate tableaux

A content system in type A

Take $\ell = 1$ and $I = \mathbb{Z}/e\mathbb{Z} = \{0, 1, \dots, e-1\}$ and *any* $\Lambda \in P^+$

- Type $A_{e-1}^{(1)}$ and define $\mathrm{c}(a) = ax^2$ and $\mathrm{r}(a) = a + e\mathbb{Z} \in I$

a	-1	0	1	$\dots$	$e{-}1$	e	$e{+}1$	$\dots$	$2e{-}1$	$\dots$
$\mathrm{r}(a)$	$e{-}1$	0	1	$\dots$	$e{-}1$	0	1	$\dots$	$e{-}1$	$\dots$
$\mathrm{c}(a)$	$-x^2$	0	x^2	$\dots$	$(e{-}1)x^2$	ex^2	$(e{+}1)x^2$	$\dots$	$(2e{-}1)x^2$	$\dots$

Take $e = 3$ and $\lambda = (4, 3, 2)$:

c

0	x^2	$2x^2$	$3x^2$	$4x^2$
$-x^2$	0	x^2		
$-2x^2$	$-x^2$			

r

0	1	2	0
2	0	1	
1	2		

$\Longrightarrow$ Setting $x = 1$ recovers the classical content function for the symmetric groups

Another content system in type A

- Type $A_{e-1}^{(1)}$ and define $\mathrm{c}(a) = \lfloor \frac{a}{e} \rfloor x^2$ and $\mathrm{r}(a) = a + e\mathbb{Z} \in I$

a	-1	0	1	$\dots$	$e{-}1$	e	$e{+}1$	$\dots$	$2e{-}1$	$\dots$
$\mathrm{r}(a)$	$e{-}1$	0	1	$\dots$	$e{-}1$	0	1	$\dots$	$e{-}1$	$\dots$
$\mathrm{c}(a)$	$-x^2$	0	0	$\dots$	0	x^2	x^2	$\dots$	x^2	$\dots$

Take $e = 3$ and $\lambda = (4, 3, 2)$:

c

0	0	0	x^2	x^2
$-x^2$	0	0		
$-x^2$	$-x^2$			

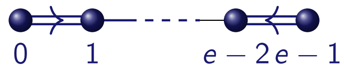
r

0	1	2	0
2	0	1	
1	2		

$\Longrightarrow$ Content systems for a given quiver are not unique

A type C content system

Take $\ell = 1$ and $I = \mathbb{Z}/e\mathbb{Z} = \{0, 1, \dots, e-1\}$

- Type $C_{e-1}^{(1)} =$ 

Define

$$\mathrm{c}(a) = \begin{cases} (a+1)^2x^4 & \text{if } a \in e\mathbb{Z} \\ (a+1)x^2 & \text{if } a \notin e\mathbb{Z} \text{ and } \lfloor \frac{a}{e} \rfloor \text{ is even} \\ -(a+1)x^2 & \text{otherwise} \end{cases}$$

and

$$\mathrm{r}(a) = \begin{cases} a + e\mathbb{Z} & \text{if } \lfloor \frac{a}{e} \rfloor \text{ is even} \\ -a + e\mathbb{Z} & \text{otherwise} \end{cases}$$

a	-1	0	1	$\dots$	$e{-}1$	e	$e{+}1$	$\dots$	$2e{-}1$	..
$\mathrm{r}(a)$	1	0	1	$\dots$	$e{-}2$	$e-1$	$e-2$	$\dots$	1	..
$\mathrm{c}(a)$	$-0x^2$	1^2x^4	$2x^2$	$\dots$	ex^2	$(e{+}1)^2x^4$	$-(e{+}2)x^2$	$\dots$	$-(2e)x^2$	..

A type C content system .../2

Take $e=3$, so type $C_2^{(1)}$, and $\lambda=(6,5,3)$:

c:

x^4	$2x^2$	3^2x^4	$-4x^2$	5^2x^4	$6x^2$	7^2x^4
0	x^4	$2x^2$	3^2x^4	$-4x^4$		
$-x^4$	0	x^4				

r:

0	1	2	1	0	1
1	0	1	2	0	
2	1	0			

In type $C_{e-1}^{(1)}$, these residue sequences can be found in the work of Ariki, Park and Speyer, which is based on earlier work of Premat

In this way, content systems define content and residue sequences

$\mathsf{c}(\mathfrak{t})$ and $\mathsf{r}(\mathfrak{t})$ for standard tableaux $\mathfrak{t}\in\mathrm{Std}(\mathcal{P}_{\ell,n})$

Content systems and seminormal forms

Fix a content system (c,r) with values in $\mathbb{k}[x]$ and let $\mathrm{R}_n^\Lambda(\mathbb{K}[x^\pm])$ be the corresponding cyclotomic KLR algebra over $\mathbb{K}[x^\pm]$

Proposition (Evseev-M.)

The cyclotomic KLR algebra $\mathrm{R}_n^\Lambda(\mathbb{K}[x^\pm])$ has a unique irreducible (graded) representation $V_{\boldsymbol\lambda}$ with basis $\{\,v_\mathfrak{t}\mid\mathfrak{t}\in\mathrm{Std}(\boldsymbol\lambda)\,\}$ such that

$$1_\mathbf{i}v_\mathfrak{t}=\delta_{\mathbf{i}\mathsf{r}(\mathfrak{t})}v_\mathfrak{t},\quad y_kv_\mathfrak{t}=\mathsf{c}_k(\mathfrak{t})v_\mathfrak{t}\quad\text{and}\quad\psi_kv_\mathfrak{t}=\beta_k(\mathfrak{t})v_\mathfrak{s}+\frac{1}{\mathsf{c}_k(\mathfrak{t})-\mathsf{c}_{k+1}(\mathfrak{t})}v_\mathfrak{t}$$

where $\mathfrak{s}=(r,r+1)\mathfrak{t}$ and $\beta_k(\mathfrak{t})\in\mathbb{K}[x^\pm]$ are prescribed scalars

- Gives all irreducible $\mathrm{R}_n^\Lambda(K[x^\pm])$-modules, for $\boldsymbol\lambda$ an ℓ-partition of n
- The proof is by checking the KLR relations
- The degree of the homogeneous basis elements $v_\mathfrak{t}$ is not very meaningful because multiplication by x^k arbitrarily shifts the grading
- If $\mathfrak{t}\in\mathrm{Std}(\boldsymbol\lambda)$ then $F_\mathfrak{t}=\prod_k\prod_\mathfrak{s}\frac{y_k-\mathsf{c}_k(\mathfrak{s})}{\mathsf{c}_k(\mathfrak{t})-\mathsf{c}_k(\mathfrak{s})}\in\mathrm{R}_n^\Lambda(\mathbb{K}[x^\pm])$ is a primitive idempotent
- Content systems always exist in types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$
- The module $V_{\boldsymbol\lambda}$ is not an irreducible $\mathrm{R}_n^\Lambda(\mathbb{k}[x])$-module

Tableaux combinatorics

Let $\mathcal{P}_{\ell,n}=\{\,(\lambda^{(1)}|\dots|\lambda^{(\ell)})\mid\sum_l|\lambda^{(l)}|=n\,\}$ be the ℓ-partitions of n

Then $\mathcal{P}_{\ell,n}$ is a poset under dominance where $\boldsymbol\lambda\trianglerighteq\boldsymbol\mu$ if

$$\sum_{l=1}^{t-1}|\lambda^{(t)}|+\sum_{r=1}^{s}\lambda_r^{(t)}\ge\sum_{l=1}^{t-1}|\mu^{(t)}|+\sum_{r=1}^{s}\mu_r^{(t)},\qquad\text{for all } s,t\ge0$$

Dominance induces a partial order on standard tableaux where $\mathfrak{s}\triangleright\mathfrak{t}$ if

$\mathrm{Shape}(\mathfrak{s}_{\downarrow m})\trianglerighteq\mathrm{Shape}(\mathfrak{t}_{\downarrow m})$ for $1\le m\le n$.

$\implies$ There exist $\mathfrak{t}^\triangleleft_{\boldsymbol\lambda},\mathfrak{t}^\triangleright_{\boldsymbol\lambda}\in\mathrm{Std}(\boldsymbol\lambda)$ such that $\mathfrak{t}^\triangleright_{\boldsymbol\lambda}\trianglerighteq\mathfrak{t}\trianglerighteq\mathfrak{t}^\triangleleft_{\boldsymbol\lambda}$, for $\mathfrak{t}\in\mathrm{Std}(\boldsymbol\lambda)$

Example If $\lambda=(4,3,2)$ then

$\mathfrak{t}^\triangleleft_\lambda=$

1	4	7	9
2	5	8	
3	6		

and $\mathfrak{t}^\triangleright_\lambda=$

1	2	3	4
4	5	6	
8	9		

Define permutations $d^\triangleleft_\mathfrak{t},d^\triangleright_\mathfrak{t}\in\mathfrak{S}_n$ by $\mathfrak{t}=d^\triangleleft_\mathfrak{t}\mathfrak{t}^\triangleleft_{\boldsymbol\lambda}=d^\triangleright_\mathfrak{t}\mathfrak{t}^\triangleright_{\boldsymbol\lambda}$

Integral bases

Fix reduced expressions $w=s_{a_1}\dots s_{a_k}$ for all $w\in\mathfrak{S}_n$ and define

$$\psi_w=\psi_{a_1}\dots\psi_{a_k}\in\mathrm{R}_n^\Lambda(\mathbb{k}[x])$$

Definition

Let $\mathfrak{s},\mathfrak{t}\in\mathrm{Std}(\boldsymbol\lambda)$, for $\boldsymbol\lambda\in\mathcal{P}_{\ell,n}$. Define

$$\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}}=\psi_{d^\triangleleft_\mathfrak{s}}y^\triangleleft_{\boldsymbol\lambda}1_{\mathbf{i}^\triangleleft_{\boldsymbol\lambda}}\psi^*_{d^\triangleleft_\mathfrak{t}}\quad\text{and}\quad\psi^\triangleright_{\mathfrak{s}\mathfrak{t}}=\psi_{d^\triangleright_\mathfrak{s}}y^\triangleright_{\boldsymbol\lambda}1_{\mathbf{i}^\triangleright_{\boldsymbol\lambda}}\psi^*_{d^\triangleright_\mathfrak{t}}$$

where $\mathbf{i}^\triangleleft_{\boldsymbol\lambda}=\mathsf{r}(\mathfrak{t}^\triangleleft_{\boldsymbol\lambda})$, $\mathbf{i}^\triangleright_{\boldsymbol\lambda}=\mathsf{r}(\mathfrak{t}^\triangleright_{\boldsymbol\lambda})$ and

$$y^\triangleleft_{\boldsymbol\lambda}=\prod_{m=1}^n\prod_{A\in\mathrm{Add}^\triangleleft_m(\mathfrak{t}^\triangleleft_{\boldsymbol\lambda})}(y_m-\mathsf{c}(A))\quad\text{and}$$

$$y^\triangleright_{\boldsymbol\lambda}=\prod_{m=1}^n\prod_{A\in\mathrm{Add}^\triangleright_m(\mathfrak{t}^\triangleright_{\boldsymbol\lambda})}(y_m-\mathsf{c}(A))$$

Importantly, $\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}},\psi^\triangleright_{\mathfrak{s}\mathfrak{t}}\in\mathrm{R}_n^\Lambda(\mathbb{k}[x])$

$\implies$ $\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}},\psi^\triangleright_{\mathfrak{s}\mathfrak{t}}\in\mathscr{R}_n^\Lambda(\mathbb{k})\cong\mathrm{R}_n^\Lambda(\mathbb{k})$ when we specialise $x=0$,

Set $f^\triangleleft_{\mathfrak{s}\mathfrak{t}}=F_\mathfrak{s}\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}}F_\mathfrak{t}$ and $f^\triangleright_{\mathfrak{s}\mathfrak{t}}=F_\mathfrak{s}\psi^\triangleright_{\mathfrak{s}\mathfrak{t}}F_\mathfrak{t}$, which are elements of $\mathrm{R}_n^\Lambda(\mathbb{K}[x^\pm])$

Cellularity over $\mathbb{K}$

Theorem

Suppose that (c, r) is a content system. Then the algebra $R_n^\Lambda(\mathbb{K}[x^\pm])$ is a graded $\mathbb{K}[x^\pm]$-cellular algebra with cellular bases:

- $\{ f_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ f_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$

Moreover, if $\boldsymbol{\lambda} \in \mathcal{P}_{\ell,n}$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathcal{P}_{\ell,n})$ then

$$V_{\boldsymbol{\lambda}} \cong R_n^\Lambda(\mathbb{K}[x^\pm]) f_{\mathfrak{st}}^{\triangleleft} = R_n^\Lambda(\mathbb{K}[x^\pm]) f_{\mathfrak{st}}^{\triangleright} = R_n^\Lambda(\mathbb{K}[x^\pm]) \psi_{\mathfrak{st}}^{\triangleleft} = R_n^\Lambda(\mathbb{K}[x^\pm]) \psi_{\mathfrak{st}}^{\triangleright}$$

This recovers almost everything from the semisimple representation theory of the symmetric group in the representation theory of $R_n^\Lambda(\mathbb{K}[x^\pm])$.

Everything is graded but the grading is not so important because $\mathbb{K}[x^\pm]$ is a graded field

Although this is cute, what we really want are cellular bases for the non-semisimple algebras $R_n^\Lambda(\Bbbk[x])$, which will give cellular bases for $\mathscr{R}_n^\Lambda(\Bbbk)$

Integral cellularity

Theorem

Suppose that (c, r) is a content system. Then the algebra $R_n^\Lambda(\Bbbk[x])$ is a graded $\Bbbk[x]$-cellular algebra with graded cellular bases:

- $\{ \psi_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$

The trickiest part of the proof is showing that these bases span the algebra over $\Bbbk[x]$. The key idea is to argue by induction on the cell poset $\mathcal{P}_{\ell,n}$

Corollary

For any domain $\Bbbk$, the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda(\Bbbk)$ of types $A_{e-1}^{(1)}$ and $C_{e-1}^{(1)}$ are graded cellular algebras with graded cellular bases $\{\psi_{\mathfrak{st}}^{\triangleleft}\}$ and $\{\psi_{\mathfrak{st}}^{\triangleright}\}$

Other cellular bases constructed by Hu-M., Bowman and Tubbenhauer-M.

The cell modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk[x])$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk[x])$ defined by the $\psi^{\triangleleft}$ and $\psi^{\triangleright}$ bases are analogues of the graded Specht modules for $R_n^\Lambda(\Bbbk[x])$

$\rightsquigarrow$ Gives graded Specht modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk)$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk)$ for $\mathscr{R}_n^\Lambda(\Bbbk)$

Categorification

Recall that $U_q(\Gamma)$ is the Kac-Moody algebra associated with the quiver Γ and let $U_{\mathcal{A}}$ be its integral form, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$

For $\Lambda \in P^+$ let $L(\Lambda)$ be the corresponding integrable highest weight module

Let $\mathrm{Rep}\, R_n^\Lambda(\mathbb{K}[x])$ be the category of finite dimensional graded $R_n^\Lambda(\mathbb{K}[x])$-modules and let $\mathrm{Proj}\, R_n^\Lambda(\mathbb{K}[x])$ be the full subcategory of projective modules. Let $[\mathrm{Rep}\, R_n^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Proj}\, R_n^\Lambda(\mathbb{K}[x])]$ be the corresponding Grothendieck groups and set

$$[\mathrm{Rep}\, R_\bullet^\Lambda(\mathbb{K}[x])] = \bigoplus_{n\ge0} [\mathrm{Rep}\, R_n^\Lambda(\mathbb{K}[x])] \quad \text{and}$$
$$[\mathrm{Proj}\, R_\bullet^\Lambda(\mathbb{K}[x])] = \bigoplus_{n\ge0} [\mathrm{Proj}\, R_n^\Lambda(\mathbb{K}[x])]$$

The graded branching rules for $R_n^\Lambda(\mathbb{K}[x])$ show that there is a natural action of $U_{\mathcal{A}}$ on $[\mathrm{Rep}\, R_\bullet^\Lambda(\mathbb{K}[x])]$ and $[\mathrm{Proj}\, R_\bullet^\Lambda(\mathbb{K}[x])]$, which is essentially given by graded induction and graded restriction

Theorem

As $U_{\mathcal{A}}$-modules, $L(\Lambda)_{\mathcal{A}} \cong [\mathrm{Proj}\, R_\bullet^\Lambda(\mathbb{K}[x])]$ and $L(\Lambda)_{\mathcal{A}}^* \cong [\mathrm{Rep}\, R_\bullet^\Lambda(\mathbb{K}[x])]$

There are two variants of this result, for the $\psi^{\triangleleft}$ and $\psi^{\triangleright}$ bases

Specht modules and simple modules

Let K be a $\Bbbk[x]$-module. We have Specht modules over K:

$$S_{\boldsymbol{\lambda}}^{\triangleleft}(K) = K \otimes_{\Bbbk[x]} S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk[x]) \text{ and } S_{\boldsymbol{\lambda}}^{\triangleright}(K) = K \otimes_{\Bbbk[x]} S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk[x])$$

The cellular bases of $R_n^\Lambda(\Bbbk[x])$ induce bilinear forms on these modules. Set

$$\mathrm{rad}\, S_{\boldsymbol{\lambda}}^{\triangleleft}(K) = \{ a \in S_{\boldsymbol{\lambda}}^{\triangleleft}(K) \mid \langle a, b \rangle_{\triangleleft} = 0 \text{ for all } b \in S_{\boldsymbol{\lambda}}^{\triangleleft}(k) \}$$
$$\mathrm{rad}\, S_{\boldsymbol{\lambda}}^{\triangleright}(K) = \{ a \in S_{\boldsymbol{\lambda}}^{\triangleright}(K) \mid \langle a, b \rangle_{\triangleright} = 0 \text{ for all } b \in S_{\boldsymbol{\lambda}}^{\triangleright}(k) \}$$

and define $D_{\boldsymbol{\mu}}^{\triangleleft}(K) = S_{\boldsymbol{\mu}}^{\triangleleft}(K) / \mathrm{rad}\, S_{\boldsymbol{\mu}}^{\triangleleft}(K)$ and $D_{\boldsymbol{\mu}}^{\triangleright}(K) = S_{\boldsymbol{\mu}}^{\triangleright}(K) / \mathrm{rad}\, S_{\boldsymbol{\mu}}^{\triangleright}(K)$

Up to shift, $\mathbb{K}$ is the unique irreducible $\mathbb{K}[x]$-module

Theorem

Up to shift, $\{ D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}) \mid D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}) \neq 0 \}$ and $\{ D_{\boldsymbol{\mu}}^{\triangleright}(\mathbb{K}) \mid D_{\boldsymbol{\mu}}^{\triangleright}(\mathbb{K}) \neq 0 \}$ are both complete sets of pairwise non-isomorphic irreducible graded $R_n^\Lambda(\mathbb{K}[x])$-modules

Using canonical bases, we obtain a criteria for when $D_{\boldsymbol{\mu}}^{\triangleleft}(\mathbb{K}) \neq 0$, and when $D_{\boldsymbol{\mu}}^{\triangleright}(\mathbb{K}) \neq 0$, and generalisations of the modular branching rules

Crystal theory gives a Mullineux map such that $D_{\boldsymbol{\mu}}^{\triangleright}(\mathbb{K}) \cong D_{\mathrm{m}(\boldsymbol{\mu})}^{\triangleleft}(\mathbb{K})$

Decomposition matrices

The theory of cellular algebras says that the graded decomposition matrices $\mathrm{D} = (d_{\boldsymbol{\lambda\mu}}(q))$ are unitriangular, where $d_{\boldsymbol{\lambda\mu}}(q) = \sum_{d\in\mathbb{Z}} [S_{\boldsymbol{\lambda}} : q^d D_{\boldsymbol{\mu}}]\, q^d$

In type $A^{(1)}_{e-1}$, (Ariki and) Brundan and Kleshchev proved that the images of the simple $\mathscr{R}^\Lambda_n(\mathbb{C})$-modules in $L(\Lambda)$ coincide with the dual canonical basis $\implies$ $d_{\boldsymbol{\lambda}\mu}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{N}[q]$

It is natural to conjecture that the same is true in type $C^{(1)}_{e-1}$

In general, in type $C^{(1)}_{e-1}$ the dual canonical basis does not coincide with basis of simple modules, as shown by the decomposition matrix for the principal block of $\mathscr{R}^{\Lambda_0}_8(\mathbb{Q})$ for the quiver $C^{(1)}_2$

Computed by Chung, Speyer and M.

	8	$7,1$	$6,1^2$	$5,1^3$	4^2	$4,3,1$
8	1					
$7,1$	q	1				
$6,1^2$	q	q^2	1			
$5,1^3$	q^2	.	q	1		
4^2	.	q	.	.	1	
$4,3,1$	q^2+1	q^3	q	.	q^2	1
$4,2^2$	$2q$	.	q^2	.	.	q
$4,2,1^2$	$2q^2$	.	q^3+q	q^2	.	q^2
$4,1^4$	q^2	.	q^3	q^4	.	.
$3^2,2$	$2q^2$	.	.	.	.	q^2
$3^2,1^2$	$2q^3$	.	q^2	.	.	q^3
$3,2^2,1$	q^4+q^2	q	q^3	.	q^2	q^4
$3,1^5$	q^3	q^2	q^4	.	.	.
2^4	.	q^3	.	.	q^4	.
$2,1^6$	q^3	q^4	.	.	.	.
1^8	q^4	.	.	.	.	.

Symmetrising form

For $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_{\ell,n})$, there exist scalars $\gamma^{\triangleleft}_{\mathfrak{t}}, \gamma^{\triangleright}_{\mathfrak{t}} \in \mathbb{K}[x^{\pm}]$ such that

$$F_{\mathfrak{t}} = \frac{1}{\gamma^{\triangleleft}_{\mathfrak{t}}} f^{\triangleleft}_{\mathfrak{tt}} = \frac{1}{\gamma^{\triangleright}_{\mathfrak{t}}} f^{\triangleright}_{\mathfrak{tt}}$$

$\implies$ $\deg^{\triangleleft}\mathfrak{t} = \frac{1}{2}\deg\gamma^{\triangleleft}_{\mathfrak{t}}$ and $\deg^{\triangleright}\mathfrak{t} = \frac{1}{2}\deg\gamma^{\triangleright}_{\mathfrak{t}}$

$\implies$ $\deg\psi^{\triangleleft}_{\mathfrak{st}} = \frac{1}{2}\big(\deg\gamma^{\triangleleft}_{\mathfrak{s}} + \deg\gamma^{\triangleleft}_{\mathfrak{t}}\big)$ and $\deg\psi^{\triangleright}_{\mathfrak{st}} = \frac{1}{2}\big(\deg\gamma^{\triangleright}_{\mathfrak{s}} + \deg\gamma^{\triangleright}_{\mathfrak{t}}\big)$

The defect polynomial of $\boldsymbol{\lambda} \in \mathcal{P}_{\ell,n}$ is $\gamma^{\diamond}_{\boldsymbol{\lambda}} = \gamma^{\triangleleft}_{\mathfrak{t}}\gamma^{\triangleright}_{\mathfrak{t}}$, for any $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_{\ell,n})$

The defect of $\boldsymbol{\lambda}$ is $\operatorname{def}(\boldsymbol{\lambda}) = \deg(\gamma^{\diamond}_{\boldsymbol{\lambda}})$

Define a trace form $\tau_\alpha : \mathrm{R}^\Lambda_n(\Bbbk) \longrightarrow \Bbbk$ by

$$\tau_\alpha(a) = \Big(\sum_{\boldsymbol{\lambda}\in\mathcal{P}_{\ell,\alpha}} \frac{1}{\gamma^{\diamond}_{\boldsymbol{\lambda}}} \chi^{\boldsymbol{\lambda}}(a)\Big)_0$$

$\implies$ $\mathrm{R}^\Lambda_n(\Bbbk[x])$ is a graded symmetric algebra

$\implies$ $S^{\triangleleft}_{\boldsymbol{\lambda}}(K)^* \cong q^{\operatorname{def}(\boldsymbol{\lambda})} S^{\triangleright}_{\boldsymbol{\lambda}}(K)$ for any $\Bbbk[x]$-module K