Newell-Littlewood Numbers

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Based on joint work with:

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Def: GL_n is the group of invertible $n \times n$ matrices over $\mathbb{C}$.

Def: A (linear) representation of GL_n is a homomorphism $\phi : GL_n \to GL(V)$ for some d-dimensional vector space V.

Equivalently: V is a GL_n-module by $g \cdot v = \phi(g)v$.

Ex: $GL_2 \circlearrowright V = Sym^2(\mathbb{C}^2) = \mathbb{C}\text{-span}\{x^2, 2xy, y^2\}$ by change of coordinates

$$x \mapsto ax + cy \quad y \mapsto bx + dy$$

Hence

$$x^2 \mapsto (ax+cy)^2, 2xy \mapsto 2(ax+cy)(bx+dy), y^2 \mapsto (bx+dy)^2$$

and the homomorphism is defined by

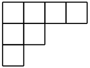
$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

Representations of GL_n II

Def: ϕ is irreducible if V has only trivial GL_n-submodules.

Theorem (I. Schur 1901)

The irred. polynomial representations ϕ_λ of GL_n are labelled by $\lambda \in Par_n = \{\text{partitions with} \leqslant n \text{ rows}\}$.

Ex: Let $n = 3$. If $\lambda = (4, 2, 1)$ then its Young diagram is .

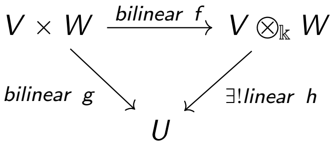
Def: The character of ϕ is $Trace(\phi(\begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix})) \in \mathbb{Z}[x_1, \ldots, x_n]$

Ex: $Tr\,\phi\left(\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) = Tr \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1 x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} = x_1^2 + x_1 x_2 + x_2^2$

Def: The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is the character of ϕ_λ.
(Our example is $s_{(2,0)}$).

Tensor products

Def: The tensor product $V \otimes_{\Bbbk} W$ of vector spaces over $\Bbbk$ is defined by



- Spanned by $v \otimes w$ subject to $(v+v') \otimes w = v \otimes w + v' \otimes w$,

  $v \otimes (w+w') = v \otimes w + v \otimes w'$, $r(v \otimes w) = (rv) \otimes w = v \otimes (rw)$.

- If V and W are GL_n-modules, so is $V \otimes_{\mathbb{C}} W$, by $g \cdot (v \otimes w) = gv \otimes gw$. Since GL_n is reductive,

$$V_\lambda \otimes_{\mathbb{C}} V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda,\mu}^{\nu}}$$

- (characters) $s_\lambda(x_1,\ldots,x_n)s_\mu(x_1,\ldots,x_n) = \sum_\nu c_{\lambda,\mu}^{\nu} s_\nu(x_1,\ldots,x_n)$.

Tensor products II

Example: $V_{(2,0)} \otimes V_{(2,0)} \cong Sym^2(\mathbb{C}^2) \otimes Sym^2(\mathbb{C}^2)$.

A basis consists of the nine tensors

$$x^2 \otimes x^2, x^2 \otimes (2xy), x^2 \otimes y^2, \ldots, y^2 \otimes x^2, y^2 \otimes (2xy), y^2 \otimes y^2.$$

GL_2 acts diagonally, e.g., $x^2 \otimes y^2 \mapsto (ax+cy)^2 \otimes (bx+dy)^2$.

The homomorphism: $\rho^{\otimes 2}(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \otimes \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix}$

$$= \begin{bmatrix} a^2 \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & 2ab \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & b^2 \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \\ ac \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & (bc+ad) \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & bd \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \\ c^2 \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & 2cd \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} & d^2 \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{bmatrix} \end{bmatrix}.$$

Example: $V_{2,0} \otimes V_{2,0} \cong Sym^2(\mathbb{C}^2) \otimes Sym^2(\mathbb{C}^2)$.

A basis consists of the nine tensors

$$x^2 \otimes x^2, x^2 \otimes (2xy), x^2 \otimes y^2, \ldots, y^2 \otimes x^2, y^2 \otimes (2xy), y^2 \otimes y^2.$$

GL_2 acts diagonally, e.g., $x^2 \otimes y^2 \mapsto (ax+cy)^2 \otimes (bx+dy)^2$.

The character: $Trace\ \rho^{\otimes 2}\left(\left[\begin{smallmatrix} x_1 & 0 \\ 0 & x_2 \end{smallmatrix}\right]\right) = Trace \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} \otimes \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix}$

$$= Trace \begin{bmatrix} x_1^2 \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} & 0 & 0 \\ 0 & x_1x_2 \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} & 0 \\ 0 & 0 & x_2^2 \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix}. \end{bmatrix}$$

$$= (x_1^2 + x_1x_2 + x_2^2)^2 = s_{(2,0)}(x_1, x_2)^2.$$

The Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$ I

The tensor product multiplicities $c_{\lambda,\mu}^{\nu}$ are known as the Littlewood-Richardson coefficients.

Theorem (Littlewood-Richardson rule)

$c_{\lambda,\mu}^{\nu}$ counts the number of semistandard Young tableaux T of shape ν/λ with μ_i-many i's that are "ballot".

Ex. Let $\lambda = \ydiagram{3,1}$, $\nu = \ydiagram{4,2,1}$, $\mu = \ydiagram{2,1}$.

What is the multiplicity of V_ν in $V_\lambda \otimes V_\mu$?
The Theorem counts LR tableaux

$$T_1 = \begin{ytableau} X & X & X & 1 \\ X & 2 \\ 1 \end{ytableau} \quad \text{and} \quad T_2 = \begin{ytableau} X & X & X & 1 \\ X & 1 \\ 2 \end{ytableau} \text{ but not } B = \begin{ytableau} X & X & X & 2 \\ X & 1 \\ 1 \end{ytableau}$$

$$\implies c_{(3,1),(2,1)}^{(4,2,1)} = 2.$$

The Littlewood-Richardson coefficients $c^{\nu}_{\lambda,\mu}$ II

Question

Characterize when $c^{\nu}_{\lambda,\mu} > 0$.

Fact: If $c^{\nu}_{\lambda,\mu} > 0$ and $c^{\gamma}_{\alpha,\beta} > 0 \implies c^{\nu+\gamma}_{\lambda+\alpha,\mu+\beta} > 0$ (semigroup)

Def: The <u>LR-semigroup</u> is

$$LR_n = \{(\lambda, \mu, \nu) \in Par_n^3 : c^{\nu}_{\lambda,\mu} > 0\}.$$

Def: The <u>saturated LR-semigroup</u> is

$$LRsat_n = \{(\lambda, \mu, \nu) \in (Par_n^{\mathbb{Q}})^3 : \exists t \in \mathbb{Q}_{>0} \text{ s.t. } c^{t\nu}_{t\lambda,t\mu} > 0\}.$$

Theorem (Knutson-Tao '99)

The LR coefficients are saturated: $LR_n = LRsat_n \cap \mathbb{Z}^{3n}$ and both generate the same rational polyhedral cone $Px \geqslant 0$.

Def: A complex valued matrix M is Hermitian if $M = {}^t\overline{M}$.

The Spectral Theorem: M is diagonalizable; has real eigenvalues.

Eigenvalue problem (19th century):

Which $(\lambda, \mu, \nu) \in (Par_n^{\mathbb{R}})^3$ occur as eigenvalues of three Hermitian $n \times n$ matrices A, B, C under the condition $A + B = C$?

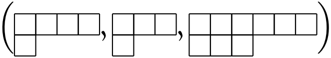
Def: The set $Eigen_n$ of such $(\lambda, \mu, \nu) \in (Par_n^{\mathbb{R}})^3$ is the eigencone.

Klyachko solved the Eigenvalue problem. One of his theorems is:

Theorem (Klyachko, '98)

$LRsat_n$ and $Eigen_n$ generate the same rational polyhedral cone $Px \geqslant 0$.

An example

$(\lambda, \mu, \nu) =$  $\in LR_2 \subseteq LRsat_2.$

Klyachko's theorem: $\exists\, A, B, C \in Hermitian_{2\times 2}$ with eigenvalues $\lambda = (41), \mu = (31), \nu = (63)$ and $A + B = C$.

After conjugating by a unitary, we may solve for a, b, c:

$$\begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} + \begin{pmatrix} 6-a & -b \\ -\overline{b} & 3-c \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.$$

Using $Trace \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} = \lambda_1 + \lambda_2$, $\det \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} = \lambda_1\lambda_2$, *etc.* reduces to two linear equations in a and c $\Longrightarrow$

$$\begin{pmatrix} \frac{11}{3} & \sqrt{\frac{8}{9}}e^{i\theta} \\ \sqrt{\frac{8}{9}}e^{-i\theta} & \frac{4}{3} \end{pmatrix} + \begin{pmatrix} \frac{7}{3} & -\sqrt{\frac{8}{9}}e^{i\theta} \\ -\sqrt{\frac{8}{9}}e^{-i\theta} & \frac{5}{3} \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}.$$

Klyachko's inequalities for $LRsat_n = Eigen_n$

Def: For $n \in \mathbb{Z}_{\geqslant 0}$, the Klyachko inequalities are

$$\sum_{k \in K} \nu_k \leqslant \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$$

for $\emptyset \subseteq I, J, K \subseteq [n]$ such that $t = \#I = \#J = \#K$ and $c^{\tau(K)}_{\tau(I),\tau(J)} > 0$, where $\tau(S) = (s_t - t, s_{t-1} - (t-1), \ldots, s_1 - 1) \in Par_n$ for $S = \{s_1 < s_2 < \ldots < s_t\} \in 2^{[n]}$.

Theorem (Klyachko '98)

$(\lambda, \mu, \nu) \in LRSat_n($"$=$"$Eigen_n) \iff (\lambda, \mu, \nu)$ satisfies Klyachko's inequalities and $|\nu| = |\lambda| + |\mu|$.

Theorem (Klyachko '98 + Knutson-Tao '99)

Horn's inequalities for $Eigen_n$ hold.

Preview of NL results

Summary of theorems (Gao-Orelowitz-Ressayre-Y. '21):

We give "Newell-Littlewood generalizations" of Klyachko's results for the classical groups SO_{2n+1}, Sp_{2n} and SO_{2n}.

Definition: SO_m are the $m \times m$ matrices M of determinant 1 that preserve a non-degenerate symmetric bilinear form $\langle \bullet, \bullet \rangle$, that is

$$\langle \vec{v}, \vec{w} \rangle = \langle M\vec{v}, M\vec{w} \rangle.$$

Sp_{2n} is defined similarly, except for a skew-symmetric bilinear form.

Theorem (Weyl '39)

For each classical group G there is a construction of representations V_λ for each $\lambda \in Par_n$. These are irreducible except for SO_{2n} when $\lambda_n \neq 0$ (in that case it is a sum of two irreducibles).

Newell-Littlewood Numbers I

Let G be classical group. Tensor products of irreps decompose:

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\lambda \in Par_n} V_\nu^{\oplus t_{\lambda,\mu}^{\nu}(G)}.$$

Definition

For $\lambda, \mu, \nu \in Par_n$, the Newell-Littlewood number is

$$N_{\lambda,\mu,\nu} = \sum_{\alpha,\beta,\gamma \in Par_n} c_{\alpha,\beta}^{\lambda} c_{\beta,\gamma}^{\mu} c_{\gamma,\alpha}^{\nu}.$$

Theorem (Koike-Terada '87)

- $t_{\lambda,\mu}^{\nu}(G) = N_{\lambda,\mu,\nu}$ *when* $\ell(\lambda) + \ell(\mu) \leqslant n$.
- *In particular, this number doesn't depend on which classical group* G $(= SO_{2n}, SO_{2n+1}, Sp_{2n})$.
- *There is a basis of symmetric functions* $\{s_{[\lambda]}\}$ *that are the "universal characters".*

Newell-Littlewood numbers II

$$N_{\lambda,\mu,\nu} = \sum_{\alpha,\beta,\gamma \in Par_n} c^{\lambda}_{\alpha,\beta} c^{\mu}_{\beta,\gamma} c^{\nu}_{\gamma,\alpha}$$

Facts from the definition:

- $|\nu| = |\lambda| + |\mu| \implies N_{\lambda,\mu,\nu} = c^{\nu}_{\lambda,\mu}$ (generalizes LR).
- If $N_{\lambda,\mu,\nu} > 0, N_{\pi,\theta,\kappa} > 0 \implies N_{\lambda+\pi,\mu+\theta,\nu+\kappa} > 0$ (semigroup).
- $N_{\lambda,\mu,\nu} > 0$ only if $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ (parity).

Def: The NL-semigroup is $NL_n = \{(\lambda,\mu,\nu) \in Par_n^3 : N_{\lambda,\mu,\nu} > 0\}$.

Def: The saturated NL-semigroup is

$$NLsat_n = \{(\lambda,\mu,\nu) \in Par_n^3 : \exists t \in \mathbb{Q}_{>0} \text{ s.t. } N_{t\lambda,t\mu,t\nu} > 0\}.$$

Conjecture: (Gao-Orelowitz-Y., 2020)

$$NL_n = NLsat_n \cap \{(x_1, x_2, \ldots, x_{3n}) \in \mathbb{Z}^{3n}_{\geqslant 0} : \sum x_i \equiv 0 (\text{mod } 2)\}.$$

$NLsat_n$ is an eigencone

Theorem (Gao-Orelowitz-Ressayre-Y., '21)

Let $\lambda, \mu, \nu \in Par_n$. Then $(\lambda, \mu, \nu) \in NLsat_n \iff$ there exist three matrices

$$M_1, M_2, M_3 \in \left\{ \begin{pmatrix} A & B \\ {}^t\bar{B} & -{}^tA \end{pmatrix} : {}^t\bar{A} = A \text{ and } {}^tB = B \right\}$$

such that $M_1 + M_2 + M_3 = 0$ which have eigenvalues $\hat{\lambda}, \hat{\mu}, \hat{\nu}$, where

$$\hat{\lambda} = (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1).$$

The set of matrices in the theorem is $\mathfrak{sp}(2n, \mathbb{C}) \cap Hermitian_{2n}$ and is derived from a theorem of [Belkale-Kumar '06] (more later).

Extended Horn/Klyachko Inequalities

Def: (Gao-Orelowitz-Y., '20) For $n \in \mathbb{N}$, the extended Horn/Klyachko inequalities are

$$\sum_{i \in I'} \lambda_i + \sum_{j \in J'} \mu_j + \sum_{k \in K'} \nu_k \leqslant \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k$$

for $I, I', J, J', K, K' \in 2^{[n]}$ s.t.

(i) $I \cap I' = J \cap J' = K \cap K' = \varnothing$;

(ii) $\#I = \#J' + \#K', \#J = \#I' + \#K', \#K = \#I' + \#J'$;

(iii) $\sum_{\alpha^1,\alpha^2,\beta^1,\beta^2,\gamma^1,\gamma^2} c^{\tau(I')}_{\alpha^1,\alpha^2} c^{\tau(J)}_{\alpha^2,\beta^1} c^{\tau(K')}_{\beta^1,\beta^2} c^{\tau(I)}_{\beta^2,\gamma^1} c^{\tau(J')}_{\gamma^1,\gamma^2} c^{\tau(K)}_{\gamma^2,\alpha^1} > 0$

NLsat-conjecture: (Gao-Orelowitz-Y., '20)

$(\lambda, \mu, \nu) \in NLsat_n \iff (\lambda, \mu, \nu)$ satisfy above inequalities.

Main results I

Theorem A (Gao-Orelowitz-Ressayre-Y., '21)

The *NLsat*-conjecture (and a better version) is true.

The proof uses:

Theorem B (Gao-Orelowitz-Ressayre-Y., '21)

For any $m \geqslant n \geqslant 1$, $NLsat_n = Sp_{2m}\text{−sat} \cap (Par_n^{\mathbb{Q}})^3$, where

$Sp_{2m}\text{−sat} = \{(\lambda, \mu, \nu) \in Par_m : \exists k \in \mathbb{Q}_{>0},\ t_{k\lambda,k\mu}^{k\nu}(Sp_{2m}) > 0\}$.

Theorem B is trivial for $m \geqslant 2n$ by definition of NL numbers, but is nontrivial for $n \leqslant m < 2n$. The argument uses [Ressayre '10] plus a dose of "Schubert calculus".

Main results II

[Belkale-Kumar '06] gives minimal inequalities for the saturated tensor cone in general type and prove an eigencone description. Thus Theorem B implies our earlier NL eigencone description and

Corollary C: (Gao-Orelowitz-Ressayre-Y., '21)

We give the first minimal set of inequalities (explicit description omitted here) for $NLsat_n$.

Corollary C to Theorem A uses a result of R. King that expresses the "6-fold NL numbers" as an LR coefficient. Hence we obtain the first "tensor product-free" (i.e., Horn-like) description of $NLsat_n$.

Theorem D: (Gao-Orelowitz-Ressayre-Y., '21); [Rough version]

NL numbers factor as LR$\times$ (smaller NL) on the boundary of $NLsat_n$.

Theorem D is an NL analogue of the LR factorization theorems of [King-Tollu-Toumazet '09] and [Derksen-Weyman '11].

Main Results III

Theorem A also shows:

Corollary E: (Gao-Orelowitz-Ressayre-Y., '21)

The NL-saturation conjecture of [Gao-Orelowitz-Y., '20] $\implies$ NL_n is also described by the extended Horn/Klyachko inequalities + the parity constraint.

In 2020, Gao-Orelowitz-Y., proves results giving evidence for the implication (true for $n = 2$; the EH/K inequalities are necessary for $N_{\lambda,\mu,\nu} > 0$; sufficient when λ, μ, or ν is a row or column).

[Gao-Orelowitz-Ressayre-Y., '21] proves (computationally) that it holds for $n \leqslant 5$.

Thank you!

Horn vs extended inequalities, $n = 2$

Horn inequalities	Extended Horn/Klyachko inequalities
$\nu_1 \leqslant \lambda_1 + \mu_1$	$\nu_1 \leqslant \lambda_1 + \mu_1$, $\lambda_1 \leqslant \mu_1 + \nu_1$, $\mu_1 \leqslant \nu_1 + \lambda_1$
$\nu_2 \leqslant \lambda_1 + \mu_2$, $\nu_2 \leqslant \lambda_2 + \mu_1$	$\nu_2 \leqslant \lambda_1 + \mu_2$, $\lambda_2 \leqslant \mu_1 + \nu_2$, $\mu_2 \leqslant \nu_1 + \lambda_2$, $\nu_2 \leqslant \lambda_2 + \mu_1$, $\lambda_2 \leqslant \mu_2 + \nu_1$, $\mu_2 \leqslant \nu_2 + \lambda_1$
$\lvert\nu\rvert = \lvert\lambda\rvert + \lvert\mu\rvert$,	$\lvert\nu\rvert \leqslant \lvert\lambda\rvert + \lvert\mu\rvert$, $\lvert\lambda\rvert \leqslant \lvert\mu\rvert + \lvert\nu\rvert$, $\lvert\mu\rvert \leqslant \lvert\nu\rvert + \lvert\lambda\rvert$
	$\lambda_1 + \mu_2 \leqslant \lambda_2 + \mu_1 + \lvert\nu\rvert$, $\mu_1 + \nu_2 \leqslant \mu_2 + \nu_1 + \lvert\lambda\rvert$ $\nu_1 + \lambda_2 \leqslant \nu_2 + \lambda_1 + \lvert\mu\rvert$, $\lambda_1 + \nu_2 \leqslant \lambda_2 + \nu_1 + \lvert\mu\rvert$ $\mu_1 + \lambda_2 \leqslant \mu_2 + \lambda_1 + \lvert\nu\rvert$, $\nu_1 + \mu_2 \leqslant \nu_2 + \mu_1 + \lvert\lambda\rvert$

Papers presented

[1] Shiliang Gao, Gidon Orelowitz, and Alexander Yong, *Newell-Littlewood numbers*, Trans. Amer. Math. Soc. 374 (2021), 6331–6366.

[2] Shiliang Gao, Gidon Orelowitz, and Alexander Yong, *Newell-Littlewood numbers II: extended Horn inequalities*, Algebr. Comb. 5 (2022), no. 6, 1287–1297.

[3] Shiliang Gao, Gidon Orelowitz, Nicolas Ressayre, and Alexander Yong, *Newell-Littlewood numbers III: eigencones and GIT-semigroups*, arXiv:2107.03152.