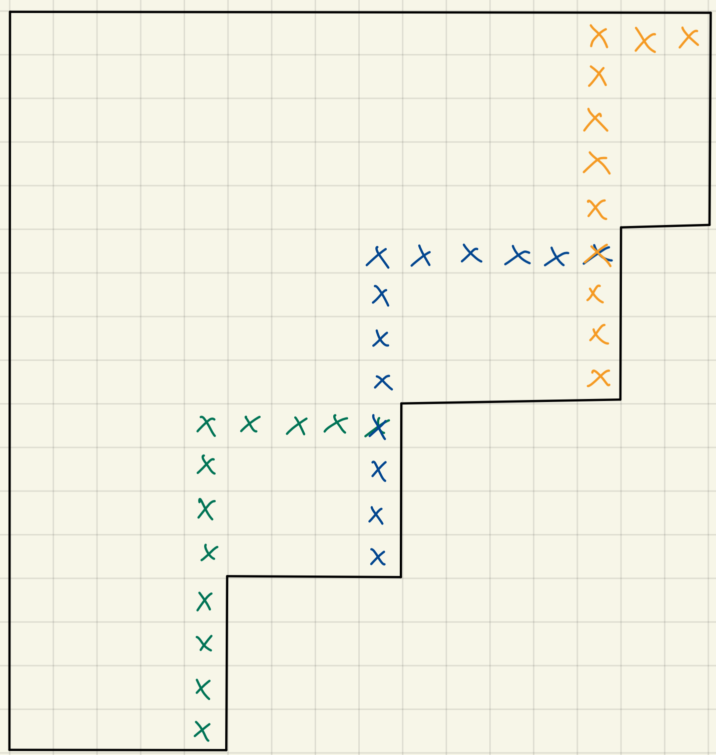
Irreducible restrictions from symmetric groups to subgroups

(OIST, March 2021)

§1. Introduction.

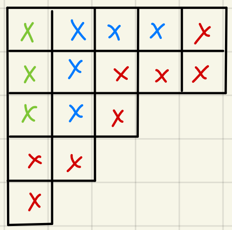
- $\mathbb{F}$ algebraically closed field of characteristic $p \geq 0$
- D^λ irreducible $\mathbb{F}S_n$-module corresponding to a p-regular partition λ of n

Theorem 1. (Jantzen-Seitz '92, K'94) $D^\lambda\downarrow_{S_{n-1}}$ is irreducible $\Leftrightarrow$ λ is a JS partion, i.e. p divides all hooks as shown:

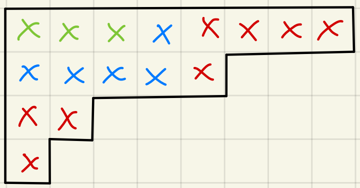


<u>Theorem 2.</u> (K.'96, Ford – K.'97, Bessenrodt-Olsson '98, ...) $D^{\lambda} \otimes \mathrm{sign} \cong D^{M(\lambda)}$, where M is the Mullineux involution.

For example, for $p=5$



$$\longrightarrow \begin{pmatrix} 8 & 5 & 3 \\ 5 & 3 & 3 \end{pmatrix} \longmapsto \begin{pmatrix} 8 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix} \longrightarrow$$



Theorem 3. (Gow–K.'99, Bessenrodt–K.'00, Graham–James'00, Morotti'18). Suppose $\dim D^\lambda$, $\dim D^\mu > 1$, and $D^\lambda \otimes D^\mu \cong D^\nu$. Then $p=2$, $n=2m$ with m odd, and (λ,μ,ν) or (μ,λ,ν) are in

$$\left\{\Big((m+1,m-1),\ (2m-2j-1,2j+1),\ (m-j,m-j-1,j+1,j)\Big)\ \Big|\ 0 \le j < \frac{m-1}{2}\right\}.$$

Question. Could all these results be made natural parts of one big theorem/program?

Answer. Yes, they are natural parts of Aschbacher–Scott program on classification of maximal subgroups of finite classical groups.

§2. Aschbacher-Scott program.

- Meta-goal: undestand maximal subgroups in finite groups Γ or, equivalently, understand primitive permutation groups (a transitive permutation group Γ is primitive $\Leftrightarrow$ a point stabilizer subgroup is maximal in Γ).

- A theorem of Aschbacher and Scott (1985) in some sense "reduces" the problem to the case where Γ is almost quasi-simple:

$$S \leq \Gamma/Z(\Gamma) \leq \mathrm{Aut}(S) \qquad (S \text{ a simple group}).$$

For example, if $S = A_n$, we get $\Gamma \in \{A_n, S_n, \hat{A}_n, \hat{S}_n, \dots\}$

- From now on let Γ be almost quasi-simple.

- Due to work of many people (Liebeck-Praeger-Saxl (1987), Liebeck-Seitz (1990), Testerman (1988), Borovik, ...) the problem is mainly reduced to the case where $\Gamma = Cl(V)$ is a classical group of Lie type.

<u>Aschbacher's Theorem</u> (1984). Let $\Gamma = Cl(V)$ be a finite classical group with the natural module V over $\mathbb{F}$ (for example, $SL(V) \le \Gamma \le GL(V)$, $G = Sp(V)$, $G = SU(V)$, $G = SO^{(\pm)}(V)$, ...). Let $G < \Gamma$ be a maximal subgroup. Then

$$G \in \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_8 \cup \mathcal{S}$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_8$ are various "standard constructions", for example,

$\mathcal{C}_1$ = {stabilizers of non-trivial proper subspaces $U \subset V$ s.t. U is non-degenerate or totally isotropic}

⋮

$\mathcal{C}_4$ = {"tensor product subgroups": $G = Cl(V_1) \otimes Cl(V_2)$ for $V = V_1 \otimes V_2$ }

⋮

$\mathcal{C}_8$ = {classical subgroups} (e.g. $Sp(V) \subset SL(V)$).

and

$\mathcal{S}$ = {almost quasi-simple groups that act absolutely irreducibly on V}.

- Aschbacher's Theorem is a result in one direction: if $G < \Gamma = Cl(V)$ is a maximal subgroup then it is one of the following... But of course we want the converse, too! Let $H \leq \Gamma$ be one of the subgroups in $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_8 \cup \mathcal{S}$. Is it maximal? "As a rule", yes (whatever this means). "As-a-rule-yes-principle".

Obtaining the converse of Aschbacher's Theorem and thus classifying the maximal subgroups in finite classical groups is sometimes called the Aschbacher-Scott program.

- The cases $H \in \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_8$ were mostly dealt with by Kleidman-Liebeck '1990 (see also Bray-Holt-Roney-Dougal '2013).

- So we may assume that $H \in \mathcal{S}$, i.e. H is an AQS group acting on V absolutely irreducibly. If H is <u>not</u> maximal, by Aschbacher's Theorem applied again,

$$H < G \in \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_8 \cup \mathcal{S}.$$

For example, $G \in \mathcal{C}_4$ means that V is tensor decomposable, which is exactly how Theorem 3 fits into the program.

- Note that Theorem 3 has infinitely many examples of tensor decomposable irreducible V's over S_n, yet there are "few of them", and they are classifiable – this is an illustration of the above "As-a-rule-yes-principle".

- The most difficult and the most open case is when $G \in \mathcal{S}$, i.e. we have an absolutely irreducible FG-module s.t. $V\downarrow_H$ is irreducible. Note that Theorem 1 about irreducible restrictions $D^\lambda\downarrow_{S_{n-1}}$ fits right in, as does Theorem 2 because in characteristic >2 $D^\lambda\downarrow_{A_n}$ is irreducible if and only if $M(\lambda) \neq \lambda$. For $p=2$, the irreducible restrictions $D^\lambda\downarrow_A$ were described by Benson '1988. So for all p, we have the explicit class $\mathcal{P}'(n)$ of (p-regular) partitions for which $D^\lambda\downarrow_{A_n}$ is irreducible.

Irreducible Restriction Problem. Let G be an almost quasi-simple group. Describe pairs (V, H), where V is an $\mathbb{F}G$-module of dimension >1 and $H < G$ is a subgroup such that $V{\downarrow}_H$ is irreducible.

- This is more general than what Aschbacher-Scott program requires (H is arbitrary), but also more natural/beautiful.

Today I want to discuss the problem where $G = S_n$. Other relevant cases, which I will skip are:

- $G = A_n$ (done: Saxl '1987 ($p=0$), K.-Sheth '2002 ($p>3$), K.-Morotti-Tiep '2020)
- $G = \hat{S}_n, \hat{A}_n$ (done: for $p=0$ (Kleidman-Wales '1991), substantial partial results for $p>0$ (K.-Tiep '2004)).
- $G = GL_n(\mathbb{F}_q)$, $(p,q)=1$ (done: K.-Tiep '2010).

From now on, $G = S_n$.

- $\underline{p=0}$: Saxl '1987 (Stunning!)
- $\underline{p>3}$: Brundan – K. '2001
- $\underline{p=2,3}$: K – Morotti – Tiep '2020

I want to show you the main result for $p>0$ (the characteristic 0 case is recovered by taking $p>n$), and explain some steps of the proof (everything I know about S_n goes into the proof...)

Here is a "prettified version" of the Main Theorem:

Main Theorem (Pretty Version). Let $n > 25$ and exclude the cases where D^λ or $D^\lambda \otimes \mathrm{sgn}$ is the natural module $D^{(n-1,1)}$ as well as the case where $p=2$ and D^λ is the basic spin module $D^{(\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor)}$. Then the restriction of an irreducible $\mathbb{F}S_n$-module D^λ of dimension >1 to a subgroup $H \leq S_n$ is irreducible if and only if one of the following holds:

(i) $G = A_n$ and $\lambda \in \mathcal{P}'(n)$.

(ii) λ is Jantzen-Seitz, and $G = S_{n-1}$

(iii) $\lambda \in \mathcal{P}'(n)$ is Jantzen-Seitz, and $G = A_{n-1}$.

(iv) $p \neq 2$, λ or $M(\lambda)$ is $(n-2, 1^2)$, $n = 2^m$, and $G = AGL_m(2)$ embedded into S_n via its natural action on the points of $\mathbb{F}_2^m$.

(v) $p \neq 2$, λ or $M(\lambda)$ is $(n-2, 1^2)$, $n = 2^m + 1 \equiv 0 \pmod p$, and $G = AGL_m(2)$ embedded into S_{n-1} via its natural action on the points of $\mathbb{F}_2^m$.

As you can see, there are "few" exceptions ... ("As-a-rule-yes-principle".)

The case we have excluded have many more exceptions. For example, if we allow λ or $M(\lambda) = (n-1, 1)$, we get a lot of doubly transitive subgroups of S_n (and a few doubly-transitive subgroups of S_{n-1}) acting irreducibly on D^λ.

In fact, let me remind you a classical result in characteristic 0: a subgroup $G < S_n$ is irreducible on the natural complex module $D_{\mathbb{C}}^{(n-1,1)}$ if and only if G is doubly transitive on $\{1, \dots, n\}$.

The result is false in characteristic p in both directions, but one can describe the exceptions explicitly – this was mostly done a while ago, for example by Mortimer' 1980, although there was still a lot of work left, and we have discovered some new exceptions. There is now a full list of exceptions – a couple of long tables.

Table II

Irreducibility of $D^{(n-1,1)}$ over doubly transitive subgroups.

G	Degree n	Transitivity	Conditions on p
S_n	n	n	
A_n	n	$n-2$	
(†) $\mathsf{C}_r^m \trianglelefteq G \leq AGL_m(r)$, r prime	r^m	2 or 3	$p \neq r$
$PSL_d(q) \trianglelefteq G \leq P\Gamma L_d(q)$, $d \geq 3$	$\dfrac{q^d-1}{q-1}$	2	$p \nmid q$
$\mathsf{A}_7 \cong G < GL_4(2)$	15	2	$p \neq 2$
$Sp_{2m}(2),\ m \geq 3$	$2^{m-1}(2^m \pm 1)$	2	$p \neq 2$
$SL_2(q) \trianglelefteq G \leq \Sigma L_2(q)$, $2 \mid q$	$q+1$	3	
$PSL_2(q) \trianglelefteq G \leq P\Sigma L_2(q)$, $2 \nmid q$	$q+1$	2	$p \neq 2$
$PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$, $G \not\leq P\Sigma L_2(q)$, $2 \nmid q$	$q+1$	3	
${}^2B_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2B_2(q))$, $q > 2$	q^2+1	2	$p \nmid (q+1+\sqrt{2q})$
$PSU_3(q) \trianglelefteq G \leq P\Gamma U_3(q)$, $q > 2$	q^3+1	2	$p \nmid (q+1)$
${}^2G_2(q) \trianglelefteq G \leq \mathrm{Aut}({}^2G_2(q))$	q^3+1	2	$p \nmid (q+1)(q+1+\sqrt{3q})$
M_{24}	24	5	$p \neq 2$
M_{23}	23	4	$p \neq 2$
M_{22}	22	3	$p \neq 2$
M_{12}	12	5	
M_{11}	11	4	
M_{11}	12	3	$p \neq 3$
$PSL_2(11)$	11	2	$p \neq 3$
HS	176	2	$p \neq 2, 3$
Co_3	276	2	$p \neq 2, 3$

Table III
Non-serial examples of irreducible restrictions from S_n.

Case	λ or $\lambda^{\mathtt{M}}$	G	n	2-transitive on	p
(S1)	$(n-2,2)$	$SL_3(2)$	7	$\{1,\ldots,n\}$	$p=5$
		$P\Gamma L_2(8)$	9		$p\neq 2,7$
		M_{11}	11		$p\neq 3,5$
		M_{11}	12		$p=2$
		M_{12}	12		$p\neq 5$
		M_{23}	23		$p\neq 2,3$
		M_{24}	24		$p\neq 2$
(S2)	$(n-2,2)$	M_{11}	12	$\{1,\ldots,n-1\}$	$p=2$
		M_{12}	13		$p=11$
		M_{23}	24		$p=11$
		M_{24}	25		$p=23$
(S3)	$(n-2,1^2)$	S_5	6	$\{1,\ldots,n\}$	$p=3$
		M_{11}	11		$p\neq 2,11$
		M_{11}	12		$p\neq 2,3$
		M_{12}	12		$p\neq 2$
		$M_{22}, \mathrm{Aut}(M_{22})$	22		$p\neq 2$
		M_{23}	23		$p\neq 2$
		M_{24}	24		$p\neq 2$
(S4)	$(n-2,1^2)$	M_{11}	12	$\{1,\ldots,n-1\}$	$p=3$
		M_{11}	13		$p=13$
		M_{12}	13		$p=13$
		$M_{22}, \mathrm{Aut}(M_{22})$	23		$p=23$
		M_{23}	24		$p=3$
		M_{24}	25		$p=5$
(S5)	$(14,1^2)$	$\mathsf{C}_2^4 \rtimes \mathsf{A}_7$	16	$\{1,\ldots,16\}$	$p\neq 2$
(S6)	$(15,1^2)$	$\mathsf{C}_2^4 \rtimes \mathsf{A}_7$	17	$\{1,\ldots,16\}$	$p=17$
(S7)	$(5,3)$	$AGL_3(2)$	8	$\{1,\ldots,8\}$	$p=5$
(S8)	$(6,3)$	$AGL_3(2)$	9	$\{1,\ldots,8\}$	$p=5$
(S9)	$(21,2,1)$	M_{24}	24	$\{1,\ldots,24\}$	$p\neq 2,3$
(S10)	$(21,1^3)$	M_{24}	24	$\{1,\ldots,24\}$	$p\neq 2,3$
(S11)	$(22,1^3)$	M_{24}	25	$\{1,\ldots,24\}$	$p=5$
(S12)	$(3,2)$	$\mathsf{C}_5 \rtimes \mathsf{C}_4$	5	$\{1,\ldots,5\}$	$p=2$
(S13)	$(4,2)$	S_5	6	$\{1,\ldots,6\}$	$p=2$
(S14)	$(6,4)$	$\mathsf{S}_6, M_{10}, \mathrm{Aut}(\mathsf{A}_6)$	10	$\{1,\ldots,10\}$	$p=2$

Some key steps of the proof:

- <u>Reduction Theorem.</u> Let $n \geq 8$ and D^λ be an irreducible representation of $\mathbb{F}S_n$ with $\dim D^\lambda > 1$. If $G < S_n$ is a subgroup such that $D^\lambda\downarrow_G$ is irreducible, the one of the following holds:

(i) $G \leq S_{n-1}$

(ii) G is 2-transitive

(iii) $p=2$ and D^λ is basic spin

(iv) $p=2$, $n \equiv 2 \pmod 4$, $\lambda = (n-1,1)$, $G \leq S_{n/2} \wr S_2$ and $G \nleq S_{n/2} \times S_{n/2}$.

Dealing with doubly transitive groups is difficult, and requires much work, in particular, new dimension bounds to be described in the end of the talk.

Proof of Reduction Theorem for $p \neq 2$ is based on the following remarkably simple

<u>Key Lemma</u> (K.-Sheth' 2000) Let $p>2$, $n \geq 4$ and $\dim D^\lambda > 1$. Then

$$\begin{array}{ccc} \dim \mathrm{End}_{S_{n-1}}(D^\lambda\downarrow_{S_{n-1}}) & < & \dim \mathrm{End}_{S_{n-2}\times S_2}(D^\lambda\downarrow_{S_{n-2}\times S_2}). \\ \| & & \| \\ \dim \mathrm{Hom}_{\mathbb{F}S_n}(M^{(n-1,1)}, \mathrm{End}_{\mathbb{F}}(D^\lambda)) & & \dim \mathrm{Hom}_{\mathbb{F}S_n}(M^{(n-2,2)}, \mathrm{End}(D^\lambda)) \end{array}$$

For $G \leq S_n$,

$\dim (M^{(n-1,1)})^G = \#G\text{-orbits on } \{1,\dots,n\} \overset{!!}{=} o_1(G)$

$\dim (M^{(n-2,2)})^G = \#G\text{-orbits on pairs } \{i,j\} =: o_2(G)$

If the group is intransitive then $G \leq S_{\lambda_1}\times S_{\lambda_2}\times\dots$ and then (for $p>2$) it is known that $D^\lambda\downarrow_{S_{\lambda_1}\times S_{\lambda_2}\times\dots}$ is irreducible $\Rightarrow (\lambda_1,\lambda_2,\dots) = (n-1,1)$, so $G \leq S_{n-1}$

Let G be transitive but not 2-transitive. Then almost always $o_2(G) > 1 = o_1(G)$.

Suppose for simplicity that $p=0$.
Then $M^{(n-2,2)} \cong M^{(n-1,1)} \oplus D^{(n-2,2)}$,
and so $(D^{(n-2,2)})^G \neq 0$.
Moreover, by the Key Lemma,

$$\mathbb{1}_{\mathbb{F}S_n} \oplus D^{(n-2,2)} \subseteq \mathrm{End}_{\mathbb{F}}(D^\lambda),$$

$$\Downarrow$$

$$\dim \mathrm{End}_{\mathbb{F}}(D^\lambda)^G \geqslant 2$$

$$\parallel$$

$$\dim \mathrm{End}_{\mathbb{F}G}(D^\lambda)$$

Hence $D^\lambda\downarrow_G$ is not irreducible!

(Just for fun, let me explain how the same Key Lemma can be used to rule out irreducibility of $D^\lambda \otimes D^\mu$:

$$\mathrm{End}_{\mathbb{F}S_n}(D^\lambda \otimes D^\mu) = \mathrm{Hom}_{\mathbb{F}S_n}\Big(\underset{\substack{\cup\!\mid \\ \mathbb{1}_{\mathbb{F}S_n} \oplus D^{(n-2,2)}}}{\mathrm{End}_{\mathbb{F}}(D^\lambda)}, \underset{\substack{\cup\!\mid \\ \mathbb{1}_{\mathbb{F}S_n} \oplus D^{(n-2,2)}}}{\mathrm{End}_{\mathbb{F}}(D^\mu)}\Big)$$

so at least 2-dimensional!)

- Dimension Bounds.

James (1983): for $\lambda_2 + \dots + \lambda_k = m$, we have

$$\dim D^{(n-m,\lambda_2,\dots,\lambda_k)} \sim \frac{\dim D^{(\lambda_2,\dots,\lambda_k)}}{m!} n^m \qquad (\text{as } n \to \infty).$$

For $m = 1,2,3,4$, James also gives explicit lower bounds:

m	lower bound for $\dim D^{(n-m,\lambda_2,\dots,\lambda_k)}$
1	$n-2$
2	$\frac{1}{2} n(n-5)$
3	$\frac{1}{6} n^2(n-9)$
4	$\frac{1}{24} n^3(n-14)$

While the asymptotic results are very difficult to use, these lower bounds are extremely useful, and we want them for arbitrary m.

Theorem (K-Morotti-Tiep'2020). Let $m \geq 4$. Define $\delta_p = \begin{cases} 0, & p \neq 2 \\ 1, & p = 2 \end{cases}$

Then for all $n \geq p(\delta_p + m - 2)$ we have

$$\dim D^{(n-m,\lambda_2,\dots,\lambda_k)} \geq \frac{1}{m!} \prod_{i=0}^{m-1} (n - (\delta_p + i)p)$$

For example, for $p=2$: $\dim D^{(n-m,\lambda_2,\dots,\lambda_k)} \geq \frac{1}{m!}(n-2)(n-4)\cdots(n-2m)$.

This is asymptotically sharp (and turns out to be quite effective).

We also prove for $p=2$ that for all n we have $\dim D^{(n-m,\lambda_2,\dots,\lambda_k)} \geq 2^m$.