# Some properties of adjustment matrices

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# Partitions

## Definition

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{N}^m_{>0}$ be a partition of $n$.
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Let $e \in \mathbb{N}_{>1}$. We say that $\lambda$ is *e-singular* if it has $e$ parts of the same size; we call $\lambda$ *e-regular* otherwise.
$\mathcal{P}(n) :=$ set of all partitions of $n$.
$\mathcal{P}^{e\text{-sing}}(n) :=$ set of all $e$-singular partitions and
$\mathcal{P}^{e\text{-reg}}(n) := \mathcal{P}(n) \setminus \mathcal{P}^{e\text{-sing}}(n)$.

# Representation theory of the symmetric group

Let $\mathbb{F}$ be a field and $\mathfrak{S}_n$ be the symmetric group on $n$ letters.

**Theorem (Specht, 1935)**

*If* $\mathrm{char}(\mathbb{F}) = 0$,

$$\{S^\lambda \mid \lambda \in \mathcal{P}(n)\} \leftrightarrow \mathrm{Irr}(\mathbb{F}\mathfrak{S}_n).$$

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**Theorem (James, 1976)**

*If* $\mathrm{char}(\mathbb{F}) = p > 0$,

$$\{D^\lambda = S^\lambda / \mathrm{rad} S^\lambda \mid \lambda \in \mathcal{P}^{\text{p-reg}}(n)\} \leftrightarrow \mathrm{Irr}(\mathbb{F}\mathfrak{S}_n).$$

We are interested to find the *decomposition numbers* $[S^\lambda : D^\mu]$.

# A presentation of $\mathfrak{S}_n$

Let $s_i := (i, i+1)$ be the basic transposition for $1 \leq i \leq n-1$.
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- $$s_i^2 = 1, \qquad i \in \{1, \ldots, n-1\}. \tag{1}$$
- $$s_i s_j = s_j s_i, \qquad \mid i - j \mid > 1. \tag{2}$$
- $$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad i \in \{1, \ldots, n-2\}. \tag{3}$$

# The Iwahori-Hecke algebra of the symmetric group

## Definition

Let $q$ be a root of unity of $\mathbb{F}$.
The *Iwahori-Hecke algebra* $\mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n) = \mathcal{H}_n$ is the unital associative $\mathbb{F}$-algebra generated by $\{T_1, T_2, \ldots, T_{n-1}\}$ :

$$\bullet (T_i - q)(T_i + 1) = 0, \qquad i \in \{1, 2, \ldots, n-1\}. \tag{4}$$

$$\bullet T_i T_j = T_j T_i, \qquad | i - j | > 1. \tag{5}$$

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$q = 1 \Rightarrow e = \mathrm{char}(\mathbb{F})$ and $\mathcal{H}_n \cong \mathbb{F}\mathfrak{S}_n$.
We sometimes say that $\mathcal{H}_n$ is a $q$-deformation of $\mathbb{F}\mathfrak{S}_n$.

# The $q$-Schur algebra

We denote the $q$-Schur algebra by $\mathcal{S}_n = \mathcal{S}_q(n, n) = \mathrm{End}_{\mathcal{H}}(\oplus_\lambda M^\lambda)$.

$$\begin{array}{ccc}
\mathcal{S}_1(n,n) & \xrightarrow{q\text{-deformation}} & \mathcal{S}_n \\
\Big\downarrow {\scriptstyle \text{Schur functor}} & & \Big\downarrow {\scriptstyle \text{Schur functor}} \\
\mathbb{F}\mathfrak{S}_n & \xrightarrow[q\text{-deformation}]{} & \mathcal{H}_n
\end{array}$$

# Representation theory of $\mathcal{H}_n$ and $\mathcal{S}_n$

## $\mathcal{H}_n$

For each $\lambda \in \mathcal{P}(n)$, we have quantum analogues of the Specht modules, denoted $S^\lambda$.

$$\{D^\lambda = S^\lambda/\mathrm{rad}S^\lambda \mid \lambda \in \mathcal{P}^{\text{e-reg}}(n)\} \leftrightarrow \mathrm{Irr}(\mathcal{H}_n).$$

*Decomposition matrix*: $\mathcal{D}_{\mathcal{H}} := ([S^\lambda : D^\mu])_{\lambda \in \mathcal{P}(n), \mu \in \mathcal{P}^{\text{e-reg}}(n)}$.

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## $\mathcal{S}_n$

For each $\lambda \in \mathcal{P}(n)$, we may define Weyl modules, denoted $W^\lambda$.

$$\{L^\lambda = W^\lambda/\mathrm{rad}W^\lambda \mid \lambda \in \mathcal{P}(n)\} \leftrightarrow \mathrm{Irr}(\mathcal{S}_n).$$

$\mathcal{D}_\mathcal{S} := ([W^\lambda : L^\mu])_{\lambda\in\mathcal{P}(n),\mu\in\mathcal{P}(n)}$.

**Warning:** For $S^\lambda$ and $W^\lambda$, we adopt the notation of Dipper and James.

# Relation between $\mathcal{D}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{S}}$

$$\begin{aligned} \text{Schur functor}: W^\lambda &\mapsto S^\lambda. \\ : L^\lambda &\mapsto \begin{cases} D^\lambda & \text{if } \lambda \in \mathcal{P}^{\text{e-reg}}(n), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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**Theorem**

*If $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}^{\text{e-reg}}(n)$, then*

$$[W^\lambda : L^\mu] = [S^\lambda : D^\mu].$$

In other words, $\mathcal{D}_{\mathcal{H}}$ is a submatrix of $\mathcal{D}_{\mathcal{S}}$.

# Adjustment matrices

In the special case where $\mathbb{F} = \mathbb{C}$, we denote the Iwahori-Hecke algebras, $q$-Schur algebras and decomposition matrices by $\mathcal{H}_n^0$, $\mathcal{S}_n^0$, $\mathcal{D}_{\mathcal{H}}^0$ and $\mathcal{D}_{\mathcal{S}}^0$ respectively.

**Theorem (Ariki, Lascoux, Leclerc and Thibon, 1996)**

*There is a recursive algorithm for calculating $\mathcal{D}_{\mathcal{H}}^0$ and $\mathcal{D}_{\mathcal{S}}^0$.*

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**Theorem (Adjustment Matrices)**

- $\exists \mathcal{A}_{\mathcal{H}} \in GL(\mathcal{P}^{\text{e-reg}}(n), \mathbb{N})$ *such that* $\mathcal{D}_{\mathcal{H}} = \mathcal{D}_{\mathcal{H}}^0 \mathcal{A}_{\mathcal{H}}$.

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**Lemma**

*A Schur functor argument can be used to show that $\mathcal{A}_{\mathcal{H}}$ is a submatrix of $\mathcal{A}_{\mathcal{S}}$.*

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**Lemma**

*A Schur functor argument can be used to show that $\mathcal{A}_{\mathcal{H}}$ is a submatrix of $\mathcal{A}_{\mathcal{S}}$. We may refer to the $(\lambda, \mu)$-entry of the adjustment matrix as $\mathrm{adj}_{\lambda\mu}$ without confusion.*

# Abacus displays of partitions

## Definition

Take an abacus with $e$ vertical runners, numbered $0, \dots, e-1$ from left to right.
Mark positions 0,1,... on the runners inceasing from left to right along successive 'rows'.

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Mark positions 0,1,. . . on the runners inceasing from left to right along successive 'rows'.
Given $\lambda \in \mathcal{P}(n)$, we get an $e$-abacus for $\lambda$ with $r \geq l(\lambda)$ beads by placing beads at positions $\beta_i(\lambda)$ and call those positions *occupied*; we say that the other positions are *unoccupied*.

$$\beta_i(\lambda) = \begin{cases} \lambda_i + r - i, & \text{if } 1 \leq i \leq l(\lambda), \\ r - i, & \text{if } i > l(\lambda). \end{cases}$$

# Abacus displays of partitions cont'd

## $\beta$-numbers

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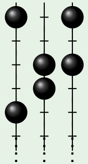
## Example

Let $e = 3$, $\lambda = (7, 6, 5^2, 1)$ and $r = 6 > l(\lambda) = 5$. Then, $(12, 10, 8, 7, 2, 0)$ is a sequence of $\beta$-number for $\lambda$.



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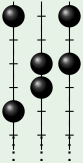

After the first unoccupied position, each bead corresponds to a part of $\lambda$. The size of that part is the number of unoccupied positions that come before it.

## $e$-cores

### Definition

Given an $e$-abacus for $\lambda$, we obtain its *e-core* by moving all the beads as high as possible on their runners.
If the $e$-core of $\lambda$ is a partition of $n - we$, we say that $\lambda$ has *e-weight* $w$.

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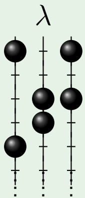

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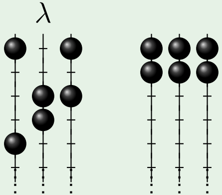


Thus, $\lambda$ has $e$-weight equal to 8.

# Blocks of $\mathcal{H}_n$ and $\mathcal{S}_n$

**Theorem (Nakayama Conjecture, Dipper and James (1987), James and Mathas (1997))**

*Let $\lambda, \mu \in \mathcal{P}(n)$. Then, $W^\lambda$ and $W^\mu$ lie in the same block of $\mathcal{S}_n$ if and only if $\lambda$ and $\mu$ have the same e-core (and the same e-weight).*

**Theorem**

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Notation:

- $\lambda \in B$ if $W^\lambda$ (or $S^\lambda$) lies in $B$.
- $e$-weight and $e$-core of $B$ = $e$-weight and $e$-core of $\lambda$

If $\lambda$ and $\mu$ lie in different blocks, then $[W^\lambda : L^\mu] = 0 = \mathrm{adj}_{\lambda\mu}$.
Therefore, the $\mathcal{D}_\mathcal{S}$ and $\mathcal{A}_\mathcal{S}$ may be partitioned into blocks.

# James's Conjecture

**Theorem ($\mathcal{D}_{\mathcal{S}}$ and $\mathcal{A}_{\mathcal{S}}$ are unitriangular)**

*Suppose that $\lambda$ and $\mu$ are partitions lying in the same block of $\mathcal{S}_n$.*
*Let $\trianglerighteq$ be the dominance order on $\mathcal{P}(n)$.*

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**James's Conjecture, 1990**

Suppose that $\lambda$ and $\mu$ lie in a block $B$ with $e$-weight $w$. If $w < \text{char}(\mathbb{F})$, then $\text{adj}_{\lambda\mu} = \delta_{\lambda\mu}$.

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**Theorem (Low, 2021)**

*James's Conjecture holds for blocks of $\mathcal{S}_n$ of weights* 3 *and* 4.

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# Normal and conormal nodes

Display $\lambda$ on an $e$-abacus with $r \geq l(\lambda)$ beads. Let $i' \equiv i + r (\text{mod } e)$.
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$\epsilon_i(\lambda) :=$ number of $i$-normal nodes.
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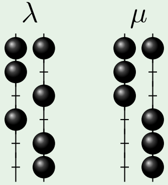
# An example

## Example

Let $e = 2$, $r = 7$, $i = 0$ (so $i' = 1$).



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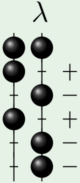

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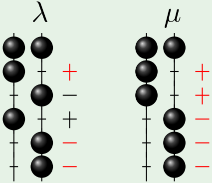


$\epsilon_0(\lambda) = 2$, $\epsilon_0(\mu) = 3$, $\varphi_0(\lambda) = 1$ and $\varphi_0(\mu) = 2$.

# Modular branching rules

Suppose that $A$ and $B$ are blocks of $\mathcal{S}_{n-k}$ and $\mathcal{S}_n$ respectively: $\mu_{\in A}$ is obtained from $\mu_{\in B}$ by moving $k$ beads on runner $i' \equiv_e i + r$ to their preceding positions.

**Theorem (Brundan, 1998. Kleshchev, 1995)**

*Suppose that $\lambda$ lies in $B$. Then,*

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- *If $\epsilon_i(\lambda) < k$, then $L^\lambda{\downarrow_A} = 0$.*

# An example of modular branching rules

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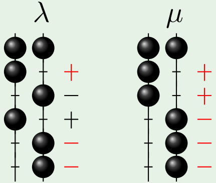


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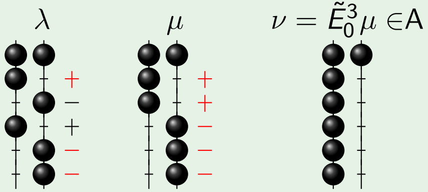


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By the modular branching rules,

- $L^\lambda\downarrow_A = 0$
- $L^\mu\downarrow_A \cong (L^\nu)^{\oplus 6}$.

# Lowerable Partitions

## Proposition (Low, 2021)

*Let $\lambda$ and $\mu$ be two distinct partitions lying in some block $B$ of $\mathcal{S}_n$. If $\epsilon_i(\lambda) < \epsilon_i(\mu)$ for some $i$, then $\mathrm{adj}_{\lambda\mu} = 0$.*

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*Suppose that $\mu$ has e-weight $w$. Then, $\tilde{E}_i^{\epsilon_i(\mu)}\mu$ and $\tilde{F}_i^{\varphi_i(\mu)}\mu$ have weight $w - \epsilon_i(\mu)\varphi_i(\mu)$.*

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We say that $(\lambda, \mu)$ is *lowerable* if there is some $e$-residue $i$ such that $\epsilon_i(\mu) > 0$, $\varphi_i(\mu) > 0$ and $\epsilon_i(\lambda) \leq \epsilon_i(\mu)$.

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If $\epsilon_i(\lambda) \leq 1$, this is equivalent to Fayers's notion of lowerable partitions in his weight 4 paper.

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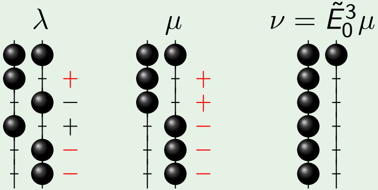


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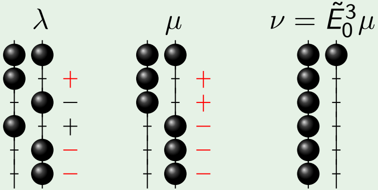


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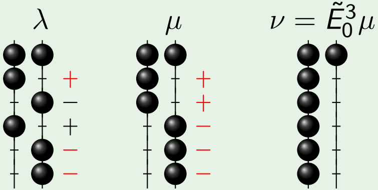


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- $\mu$ has weight 6. $\nu$ has weight $6 - 3x2 = 0$.
- $(\lambda, \mu)$ is lowerable and $\mathrm{adj}_{\lambda\mu} = 0$ by the previous proposition.

# What about $\mathrm{char}(\mathbb{F}) \leq w$?

## Theorem (Fayers, 2005)

*Let $\lambda$ and $\mu$ be two distinct partitions lying in a block $B$ of $\mathcal{H}_n$ with $w = \mathrm{char}(\mathbb{F}) = 2$. Then,*
$\mathrm{adj}_{\lambda\mu} = 0$ unless *both $\lambda$ and $\mu$ induces semi-simply to the Rouquier block.*

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**Theorem (Fayers and Tan, 2006)**

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- $\mathrm{char}(\mathbb{F}) = 3$*: both $\lambda$ and $\mu$ induces semi-simply to the Rouquier block.*
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## Theorem (Low and Tan, recent result)

*When* $\mathrm{char}(\mathbb{F}) \leq w$, *the same description holds for blocks of* $\mathcal{S}_n$ *of weights 2 and 3.*

# Links to papers

- Low, A.Y.R. Adjustment matrices for the principal block of the Iwahori-Hecke algebra $\mathcal{H}_{5e}$
  - Journal of Algebra
  - arXiv
- Low, A.Y.R. James's Conjecture holds for blocks of $q$-Schur algebras of weights 3 and 4