On Temperley-Lieb algebra at odd root of unity

Jun Murakami

Waseda University

Joint with C. Blanchet and M. de Renzi

Diagrammatic construction of representations of small quantum sl_2,
Transformation Groups **27** (2022), 751–795.

Quantum sl_2 at root of unity

- $\mathcal{U}_q(sl_2)$ $\qquad K,\ E,\ F$

$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ EF-FE = \frac{K-K^{-1}}{q-q^{-1}}$$

- Restricted $\mathcal{U}_q(sl_2)$ $\qquad q = \exp(\pi\sqrt{-1}/r)$

$$K^{2r} = 1, \quad E^r = F^r = 0$$

- Small $\mathcal{U}_q(sl_2)$ $\qquad q = \exp(2\pi\sqrt{-1}/r),\ \ q$: odd

$$K^r = 1, \quad E^r = F^r = 0$$

- $\mathcal{U}_q(sl_2)$

  Mod $\mathcal{U}_q(sl_2)$: cateroty of $\mathcal{U}_q(sl_2)$ modules generated by the natural representation $\longleftrightarrow T(m,n)$

- $\widetilde{\mathcal{U}}$: Restricted $\mathcal{U}_q(sl_2)$ $\qquad q = \exp(\pi\sqrt{-1}/r)$

  Mod $\widetilde{\mathcal{U}} \longleftarrow$ some extension of $T(m,n)$

  S. Moore, Diagrammatic morphisms between indecomposable modules of $\mathcal{U}_q(sl_2)$, Internat. J. Math. **31** (2020), 2050016.

- $\overline{\mathcal{U}}$: Small $\mathcal{U}_q(sl_2)$ $\qquad q = \exp(2\pi\sqrt{-1}/r)$, $\;q$: odd

  Mod $\overline{\mathcal{U}} \longleftarrow$ some extension of $T(m,n)$

Structure of representations

Representations coming from the tensor of tne natural representation.

- X_1 : natrural representatiion,
  $X_m \subset X_1^{\otimes m}$

- X_{r-1}

- P_m

- $X_{2r-1} \oplus P_{r-3} = P_{r-2} \otimes X_1$

Hom spaces

$$\dim_{\mathbb{C}} \mathrm{End}_{\overline{U}}(X_m) = 1 \qquad 0 \le m \le r-1,$$

$$\dim_{\mathbb{C}} \mathrm{End}_{\overline{U}}(P_m) = 2 \qquad r \le m \le 2r-2,$$

$$\dim_{\mathbb{C}} \mathrm{End}_{\overline{U}}(X_{2r-1}) = 4$$

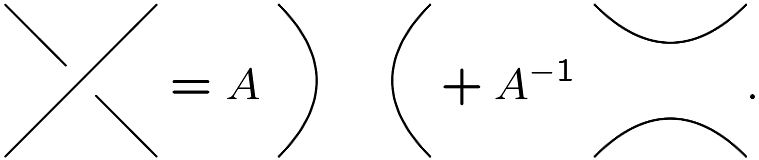
$$\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{U}}(P_m, X_{2r-m-1}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\overline{U}}(X_{2r-m-1},\ P_m) = 1 \qquad r \le m \le 2r-2,$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{U}}(P_m, P_{3r-m-2}) = 2 \qquad r \le m \le 2r-2,$$

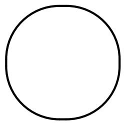
$$\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{U}}(X_{r-1}, X_{2r-1}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\overline{U}}(X_{2r-1},\ X_{r-1}) = 2.$$

Temperley-Lieb category $T(m,n)$

$T(m,m)$: the quotient of the Hecke algebra by the Kauffman bracket skein relation.



This relation implies that



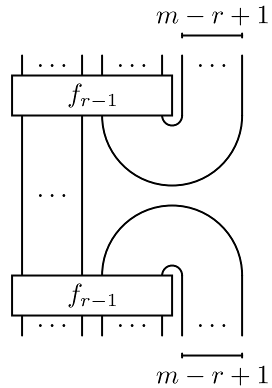
$$= -A^2 - A^{-2}.$$

Temperley-Lieb category is obtained by adding $\cup$ and $\cap$.

$A = q^{(r+1)/2}$, $q = A^2$, $\{n\} = q^n - q^{-n}$, $[n] = \dfrac{\{n\}}{\{1\}}$, $[n]! = [n][n-1]\cdots[1]$.

- Jones-Wenzl idempotent
  $$f_m = f_{m-1} + \frac{[m-1]}{[m]} f_{m-1} e_{m-1} f_{m-1}$$
  idempotent for X_m

- Idenpotent g_m for P_m $\quad (r \le m \le 2r-2)$

$$g_m = f_m + \frac{1}{[r]} \cdot h_m, \quad h_m = (-1)^m [2r-m-1]$$ 

Note that g_m is well-defined (a kind of derivation).

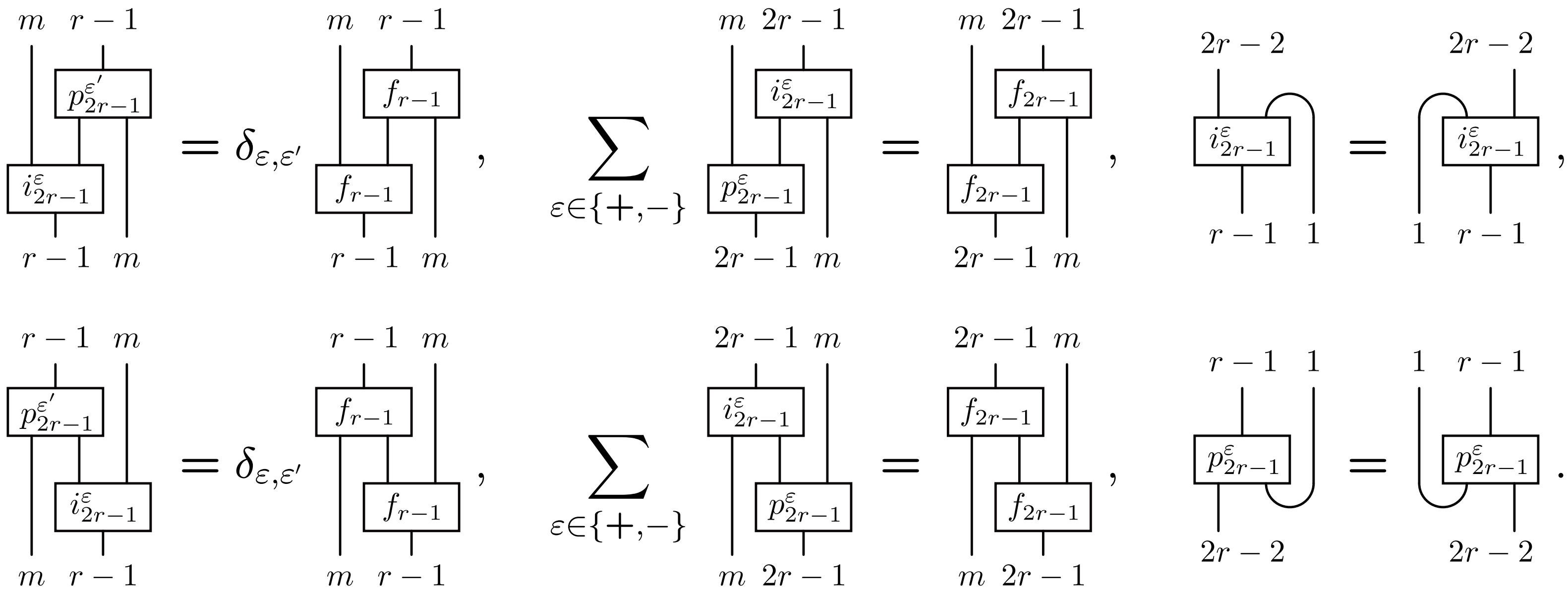
Then these correspond to the idempotents of Mod $\overline{U}$.

Missing Hom's $X_{2r-1} \cong X^-_{r-1} \oplus X^+_{r-1}, \quad X^\pm_{r-1} \cong X_{r-1}.$

There is NO hom between X_{2r-1} and X_{r-1}.

So add the following two hom's to $T(m,n)$.

$$p^\pm_{r-1} : X_{2r-1} \to X^\pm_{r-1}, \qquad i^\pm_{r-1} : X^\pm_{r-1} \to X_{2r-1}$$

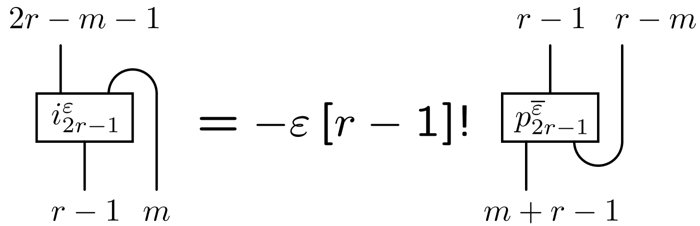


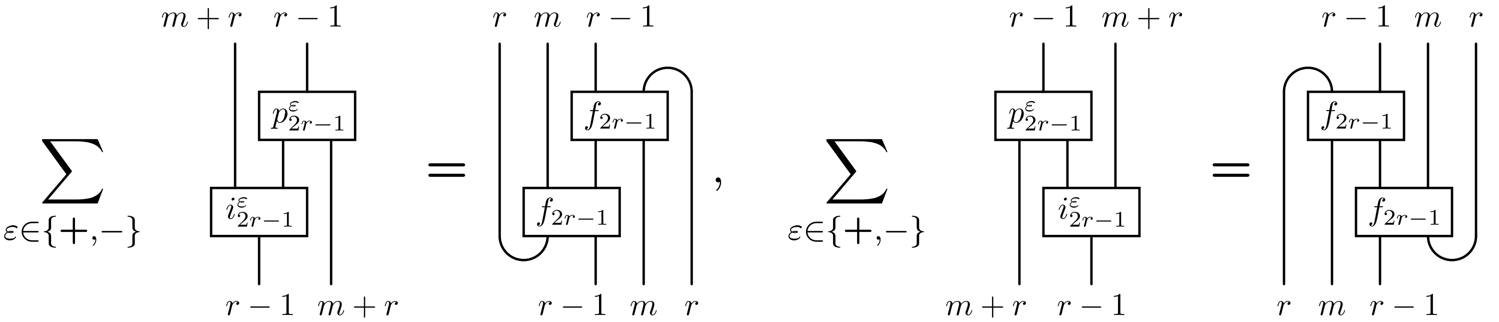
Theorem. The functor from the extended $T(m,m)$ to $\mathrm{Mod}\,\overline{U}$ is well-defined and surjective.

Problem. What is the kernel of this functor?

Additional relations

Following relations hold.





Ploblems

- Construct GOOD TQFT from the projective modules P_m.

  In usual construction, trace of a projective module is 0. On the other hand, we have non-vanishing three manifold invariant coming from pseudo trace (integral of Hopf algebra).

- sl_n case.

  sl_n skein theory is known. But its relation to the representations of restricted or small quantum groups is not clear yet (even for sl_2) if q is a root of unity.

- Understand pseudo trace for characteristic p case by 'derivation'.