SEMICUSPIDAL CATEGORIES AND MICROLOCALIZATION

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based on projects with B. Davison
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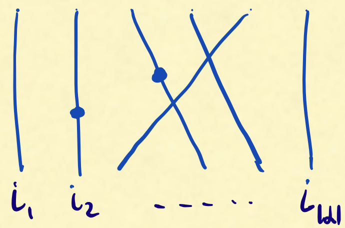
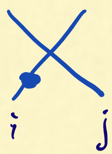
KLR ALGEBRAS

$Q=(I,E)$ quiver (finite or affine type, simply laced), $\nu \in \mathbb{Z}_{\geq 0}^{I}$ dim. vector

$R(\nu) = \Big\langle$ diagrams, $\sum i_j = \nu$ 

$\Big\rangle \Big/$ relations:



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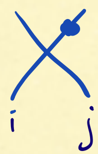
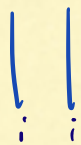
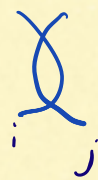
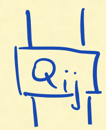


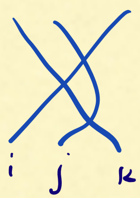
$= \delta_{ij}$


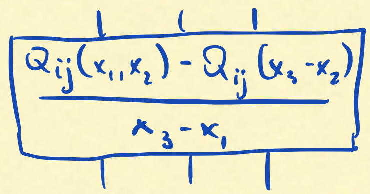


$= \begin{cases} 0 & i=j \end{cases}$

$i - j$


otherwise

$-$


$=$


if $i=k\neq j$, else 0

Thm (KLR) $\left(\coprod_{\nu} R(\nu)\text{-grpmod},\ \mathrm{Ind},\ \mathrm{Res}\right)$ categorifies $U_q(n_Q^+)$.

PBW BASES

Let Q be Dynkin. $\quad U_q(\mathfrak{n}^+) = \mathrm{span}\langle E_{\alpha_1}^{(n_1)} \cdot \ldots \cdot E_{\alpha_k}^{(n_k)} \rangle$ as a vector space,

$\alpha_1 < \ldots < \alpha_k$ convex order on positive roots

Categorified version: $\quad R(\nu)$-mod is stratified

(Brundan-Kleshchev-McNamara, Kato)

Stratas: $\forall \alpha$ positive root, $n > 0$,

$$\mathcal{C}(n\alpha) = \left\{ V \in R(n\alpha)\text{-mod} : \mathrm{Res}_{\theta,\eta} V = 0 \text{ unless } \begin{array}{l} \theta = \text{sum of roots } < \alpha \\ \eta = \text{sum of roots } > \alpha \end{array} \right\}$$

$= $ modules over $R(n\alpha)/R(n\alpha)1_{nsc}R(n\alpha)$ $\qquad$ (semicuspidal modules)

$\nwarrow$ idempotent

$\mathcal{C}(n\alpha) \simeq C(n\alpha)$-mod, $\quad C(n\alpha) \simeq \Bbbk[x_1, \ldots, x_n]$ $\qquad$ char $\Bbbk$ arbitrary

In particular, $R(\nu)$ is affine quasihereditary / $R(\nu)$-mod is affine h.w. category.

AFFINE TYPE

Let Q be of affine $ADE^{(1)}$ type.

PBW basis of $U_q(n^+)$ is more complicated, b/c imaginary roots are "incompatible".

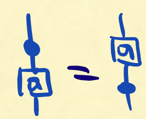
$$\left\langle E_{\beta_1}^{(n_1)} \cdots E_{\beta_k}^{(n_k)} S_\delta^{(\dots)} E_{\gamma_1}^{(m_1)} \cdots E_{\gamma_l}^{(m_l)} \right\rangle \qquad \beta_1 < \dots < \delta < \dots < \gamma_1$$

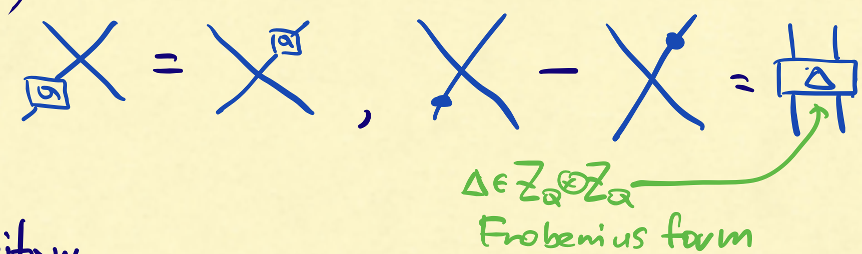
Categorified version (Kleshchev-Muth, McNamara):

- $R(\nu)$ is still stratified by $C(n\alpha)$
- when α real, $C(n\alpha) \simeq \Bbbk[x_1, \dots, x_n]$
- when $\alpha = \delta$ imaginary & char $\Bbbk = 0$, $C(n\delta) \simeq$ "affine zigzag algebra" $Z_n^{aff}(Q)$

Z_Q : zigzag algebra for the Dynkin quiver $\qquad Z_{A_1} = \Bbbk[c]/c^2$

In general, $Z_Q = \Bbbk[\bar{Q}] / $ length 3

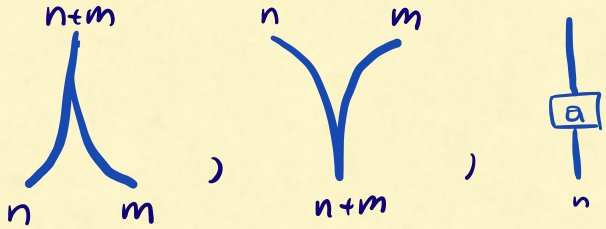
$Z_n^{aff}(Q) = \{$ diagrams with  ,  , $a \in Z_Q$,  $\}$ / S_n-relations, Z_Q-relations, 



In particular, $C(\delta)$ is not affine quasi-hereditary.

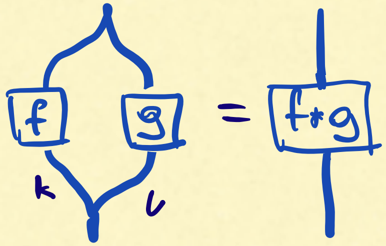
POSITIVE CHARACTERISTIC

First guess: consider the "Schur version" $S_n(Z_Q)$

generators: +  $a \in \mathrm{Sym}^n(Z_Q[x])$

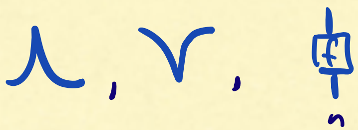
relations: hard to write down, "whatever comes from a faithful representation"

Ex



$$\boxed{\begin{gathered} f * g := \frac{1}{k!\,l!}\,\mathrm{Sym}\Big(\prod_{\substack{1\le i\le k \\ k+1\le j\le k+l}} \Big(1+\frac{\Delta_{ij}}{x_i - x_j}\Big) f \boxtimes g\Big) \\ f \boxtimes g = f(x_1,\dots,x_k)\cdot g(x_{k+1},\dots,x_{k+l}) \end{gathered}}$$

This does not work on the nose!

Conj (Maksimau - M.) Let $S_n'(Z_Q) \subset S_n(Z_Q)$ be the full rank $\mathbb{Z}$-sublattice generated by , $f \in \begin{cases} Z_Q[x], & n=1 \\ \mathrm{Sym}^n \mathbb{Z}[x], & n>1. \end{cases}$ Then $\boxed{C(n\delta)_{\mathbb{F}_p} \simeq S_n'(Z_Q) \otimes_{\mathbb{Z}} \mathbb{F}_p}$

Ex $Q = \bullet \rightrightarrows \bullet$, $Z = \mathbb{Z}[c]/c^2$. $f, g \in \mathbb{Z}[x,c]/c^2$ $\rightsquigarrow$ $f * g = f \boxtimes g + g \boxtimes f + \frac{c_1 + c_2}{x_1 - x_2}(f \boxtimes g - g \boxtimes f)$

$c * c = 2c_1c_2$, but cannot get c_1c_2

KRONECKER QUIVER CASE

$\boxed{\text{Thm (Maksimau – M., '20) Conjecture is true for } Q = \bullet \rightrightarrows \bullet}$

Rmk $\exists$ version for wKLR ; zigzag $\rightsquigarrow$ extended zigzag (M.-M. '23?)

This recovers the fact that quiver Schur in type $\hat{A}_n$ is affine qher.

$R(\alpha) \simeq H_*^{BM}(St)$, $St = \{ \text{rep. of } Q + 2 \text{ compatible full flags} \}$ (rep. dim $=\alpha$)

$St \to RepQ$ forgetful $\Rightarrow$ $R(\alpha)$ sheaf of algebras over $RepQ$

$$\mathcal{D}^b(Rep \cdot \rightrightarrows \cdot) \simeq \mathcal{D}^b(Coh\, \mathbb{P}^1) \quad \Rightarrow \quad Tor_n \mathbb{P}^1 \subset Rep_{n\delta}(\cdot \rightrightarrows \cdot)$$

Def C smooth curve $/\mathbb{C}$, $St_n = \{ \text{torsion sheaf on } C + 2 \text{ compatible full flags} \}$ (torsion sheaf length $= n$)

$H_*^{BM}(St_n) =: R_n(C)$ – KLR algebra of C

Full flags $\rightsquigarrow$ partial flags $\Rightarrow$ $S_n(C)$ Schur algebra

$$Tor_n \mathbb{P}^1 \subset Rep_{n\delta}(\cdot \rightrightarrows \cdot) \quad \Rightarrow \quad R(n\delta) \xrightarrow{\Phi} S_n(\mathbb{P}^1)$$

$Z_{A_1} = \mathbb{Z}[c]/c^2 = H^*(\mathbb{P}^1)$ $\qquad S_n(\mathbb{P}^1) \simeq S_n(Z_{A_1})$ $\qquad \mathrm{Im}\,\Phi \simeq S_n'(Z_{A_1})$

Non-surjectivity of $\Phi \Leftrightarrow H^*(Coh_n)$ is not generated by tautological classes over $\mathbb{Z}$

STRATEGY FOR OTHER TYPES

Main issue: $Z_Q \not\simeq H^*(X)$, cannot be realized as "Chriss-Ginzburg style" convolution algebra

<u>But</u>: can modify convolution to get it from

$$\begin{tikzcd} \coprod \mathbb{P}^1 \arrow[r] \arrow[rr, bend left] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathbb{C}^2/\Gamma} \arrow[d] \\ & 0 \arrow[r] & \mathbb{C}^2/\Gamma \end{tikzcd}$$

(Stroppel-Webster)

$\mathcal{D}^b(\mathrm{Rep}\,\Pi_Q) \simeq \mathcal{D}^b(\mathrm{Coh}\,\widetilde{\mathbb{C}^2/\Gamma}) \Rightarrow$ want to restrict to an open in $\mathrm{Rep}\,\Pi_Q \simeq T^*\mathrm{Rep}\,Q$

<u>Dimensional reduction (Kinjo)</u> $H^{BM}(St) \simeq H^*(T^*[-1]St, \varphi)$ vanishing cycle cpx

"Classical truncation" of $T^*[-1]St$ is $\{$rep of Π_Q + 2 compatible flags$\}$

$\Rightarrow T^*[-1]St \longrightarrow \mathrm{Rep}\,\Pi_Q$

Thm (Davison-M., '23?) This upgrades $R(\alpha)$ to a sheaf of algebras over $\mathrm{Rep}\,\Pi_Q$

Very computable via hyperbolic localization; works over $\mathbb{Z}$.

$\mathrm{Rep}_{n\delta}\Pi_Q \supset \mathrm{Tor}_n(\widetilde{\mathbb{C}^2/\Gamma})$ as semistable representations

<u>Expectation / WIP</u> $S_n(Z_Q) \simeq$ Schur algebra of torsion sheaves supported on E

$S_n'(Z_Q) \simeq$ image of restriction map

$C(n\delta) \simeq S_n'(Z_Q)$ (restriction kills singular supports of non-cuspidal idempotents)