Cyclotomic KLR algebras

Cellular bases, KLRW algebras and content systems

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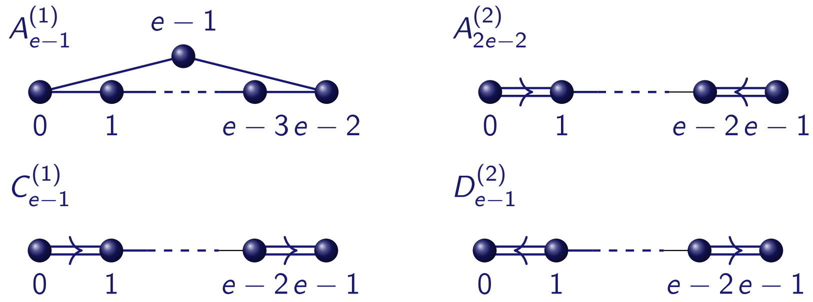
Symmetrisable Quivers

Let Γ be a symmetrisable quiver. We will focus on:



All quivers have vertex set $I=\{0,1,\dots,e-1\}$ $\qquad \mathfrak{S}_n \circlearrowleft I^n$

To the quiver Γ we attach:

- Fundamental weights $\{\Lambda_i\}$, simple roots $\{\alpha_i\}$, simple coroots $\{\alpha_i^\vee\}$
- A Cartan matrix $C=(c_{ij})_{i,j\in I}$ and bilinear form $(\alpha_i^\vee,\alpha_j)=d_ic_{ij}$
- Positive and dominant root lattices: $Q^+=\bigoplus_i \mathbb{N}\alpha_i$ and $P^+=\bigoplus_i \mathbb{N}\Lambda_i$
- A quantised Kac-Moody algebra $U_q(\mathfrak{g}_\Gamma)$
- $\mathfrak{S}_n=\langle r_1,\dots,r_{n-1}\rangle$ the symmetric group of degree n, $r_k=(k,k+1)$

Khovanov-Lauda-Rouquier algebras

The KLR algebra $\mathscr{R}_n$ is the unital associative K-algebra generated by

$$\{\,1_{\mathrm{i}}\mid \mathrm{i}\in I^n\,\}\cup\{\,\psi_k\mid 1\le k<n\,\}\cup\{\,y_k\mid 1\le k\le n\,\}$$

subject to the relations:

- $1_{\mathrm{i}}1_{\mathrm{j}}=\delta_{\mathrm{i},\mathrm{j}}1_{\mathrm{i}},\quad y_k1_{\mathrm{i}}=1_{\mathrm{i}}y_k,\quad y_ky_m=y_my_k,\quad \sum_{\mathrm{i}\in I^n}1_{\mathrm{i}}=1$
- $\psi_k1_{\mathrm{i}}=1_{r_k\mathrm{i}}\psi_k,\quad \psi_k\psi_m=\psi_m\psi_k$ if $|m-k|>1$
- $(\psi_ky_{k+1}-y_k\psi_k)1_{\mathrm{i}}=\delta_{i_k,i_{k+1}}1_{\mathrm{i}}=(y_{k+1}\psi_k-\psi_ky_k)1_{\mathrm{i}}$
- $\psi_k^21_{\mathrm{i}}=Q_{i_k,i_{k+1}}(y_k,y_{k+1})1_{\mathrm{i}}$
- $(\psi_{k+1}\psi_k\psi_{k+1}-\psi_k\psi_{k+1}\psi_k)1_{\mathrm{i}}=\delta_{i_k,i_{k+2}}\frac{Q_{i_k,i_{k+1}}(y_k,y_{k+1})-Q_{i_{k+1},i_k}(y_{k+1},y_{k+2})}{y_k-y_{k+2}}$

where $Q_{i,j}(u,v)=Q_{j,i}(v,u)$ and $Q_{i,j}(u,v)=\begin{cases}u-v & \text{if } i\to j\\ (u-v)(v-u) & \text{if } i\leftrightarrows j\\ u-v^2 & \text{if } i\Rightarrow j\\ u-v^3 & \text{if } i\Rrightarrow j\end{cases}$

Importantly, $\mathscr{R}_n$ is a $\mathbb{Z}$-graded algebra with degree function

$$\deg 1_{\mathrm{i}}=0,\quad \deg y_m1_{\mathrm{i}}=(\alpha_{i_m},\alpha_{i_m}) \text{ and } \deg\psi_k1_{\mathrm{i}}=-(\alpha_{i_k},\alpha_{i_{k+1}})$$

First steps

For $w\in\mathfrak{S}_n$ set $\psi_w=\psi_{r_{a_1}}\dots\psi_{r_{a_k}}$, where $w=r_{a_1}\dots r_{a_k}$ (reduced)

Theorem (Khovanov-Lauda, Rouquier)

The KLR algebra $\mathscr{R}_n$ is $\mathbb{Z}$-free with homogeneous basis

$$\{\,y_1^{a_1}\dots y_n^{a_n}\psi_w1_{\mathrm{i}}\mid a_k\in\mathbb{N}, w\in\mathfrak{S}_n, \mathrm{i}\in I^n\,\}$$

Linear independence is proved using a faithful polynomial representation

Recall that $U_q(\mathfrak{g}_\Gamma)$ be the quantised Lie algebra/Kac-Moody algebra associated to the symmetrisable quiver Γ

Theorem (Khovanov-Lauda, Rouquier)

The KLR algebras $\mathscr{R}_n$ categorify $U_q^-(\mathfrak{g}_\Gamma)$. More precisely, there is an I-graded bialgebra isomorphism $\mathrm{f}\to\bigoplus_{\alpha\in Q^+}[\operatorname{Rep}\mathscr{R}_\alpha]$

Cyclotomic KLR algebras

Fix a dominant weight $\Lambda \in P^+$

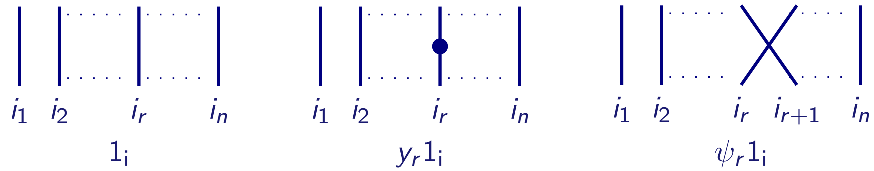
The cyclotomic KLR algebra of weight Λ and type Γ is

$$\mathscr{R}_n^\Lambda = \mathscr{R}_n / (y_1^{(\Lambda,\alpha_{i_1})} 1_{\mathbf{i}} \mid \mathbf{i} \in I^n)$$

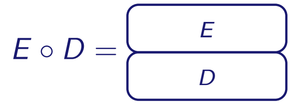
1. Type A_∞ includes the Khovanov arc algebras of Brundan-Stroppel
2. (Brundan-Kleshchev) Over a field, if Γ is of type $A_{e-1}^{(1)}$ then $\mathscr{R}_n^\Lambda$ is isomorphic to a (degenerate or non-degenerate) Ariki-Koike algebra. This includes $F\mathfrak{S}_n$ as a special case
3. (Lauda-Vazirani) The algebras $\mathscr{R}_n^\Lambda$ categorify the crystal graph of the irreducible highest weight $U_q(\mathfrak{g}_\Gamma)$-module $L(\Lambda)$
4. (Kang-Kashiwara) The algebras $\mathscr{R}_n^\Lambda$ categorify the irreducible highest weight $U_q(\mathfrak{g}_\Gamma)$-module $L(\Lambda) \cong \bigoplus_n [\operatorname{Rep} \mathscr{R}_n^\Lambda]$
5. (Hu-Shi) If $\alpha \in Q^+$ and $\mathbf{i}, \mathbf{j} \in I^n$ then
$$\dim_q 1_{\mathbf{i}} \mathscr{R}_\alpha^\Lambda 1_{\mathbf{j}} = \sum_{w \in \mathfrak{S}_{\mathbf{i},\mathbf{j}}} \prod_{t=1}^{n} q_{i_t}^{N_{1,\mathbf{i},t}^{\lambda} - 1} [N_{w,\mathbf{i},t}^\Lambda]_{i_t}, \qquad \text{for } N_{w,\mathbf{i},t}^\Lambda \in \mathbb{Z}$$
6. Until recently, bases only known in type A (Hu-M., Webster, Bowman)

A diagrammatic presentation for $\mathscr{R}_n^\Lambda$

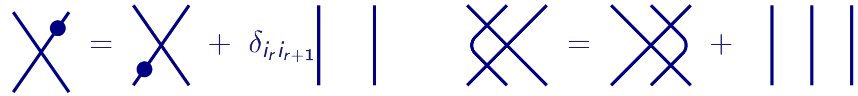
The algebra $\mathscr{R}_n^\Lambda$ has a diagrammatic presentation with generators:



Diagrams are equivalent up to isotopy. If D and E are diagrams then $D \circ E$ is zero if the residues of the strings are different and if they coincide then:



The relations become "local" operations on the diagrams that describe how to move dots and strings past crossings. For example:



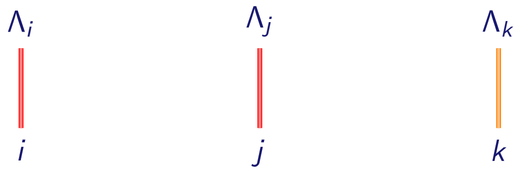
$$y_{r+1}\psi_r 1_{\mathbf{i}} = (\psi_r y_r + \delta_{i_r i_{r+1}}) 1_{\mathbf{i}} \qquad \psi_r \psi_{r+1} \psi_r 1_{\mathbf{i}} = (\psi_{r+1}\psi_r\psi_{r+1} + 1) 1_{\mathbf{i}}$$

Webster diagrams

Webster wanted to categorify tensor products of Fock spaces

To do this he introduced:

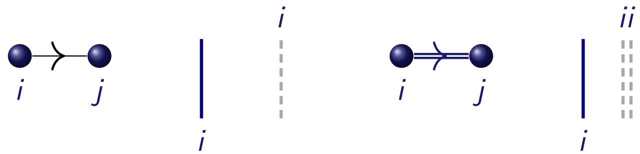
- Red strings



Affine string

The red strings generalise the cyclotomic relations in $\mathscr{R}_n^\Lambda$

- Ghost strings for each edge ϵ in the quiver with tail i:



The ghost shifts σ_ϵ can be chosen arbitrarily, but we usually fix $\sigma_\epsilon = 1$

Ghost strings have ghost dots when their solid string has dots

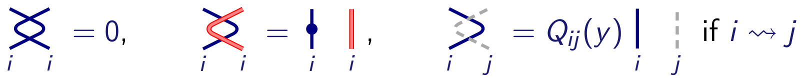
If σ_ε is small then the KLRW relations will ensure that we recover the corresponding KLR algebra

KLRW algebras

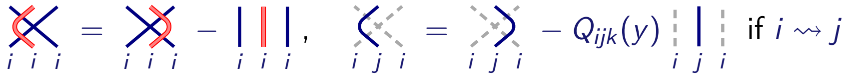
The (weighted) KLRW algebra W_n is the diagram algebra spanned by the Webster diagrams subject to the multi-local relations:

- Dots pass through crossings except for:

- Reidemeister II relations hold except for:



- Reidemeister III relations hold except for:



Together with the mirror relations — except that red strings go through ghost crossings

The algebra W_n is $\mathbb{Z}$-graded, with the grading being determined by the Cartan matrix of the quiver

A basis theorem

Fixing the end-points, each $w \in \mathfrak{S}_n$ determines a diagram $D(w)1_{\mathrm{i}}$
Applying dots, gives a dotted diagram $y_1^{a_1} \dots y_n^{a_m} D(w)1_{\mathrm{i}}$

Theorem

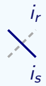
The KLRW algebra W_n is $\mathbb{Z}$-free with basis $\{y_1^{a_1} \dots y_n^{a_m} D(w)1_{\mathrm{i}}\}$

The key to proving this result is a faithful polynomial representation of W_n

Proposition

Let $P = \bigoplus_{\mathrm{i} \in I^n} \mathbb{Z}[y_1, \dots, y_n]1_{\mathrm{i}}$. Then P is a faithful W_n-module with

$1_{\mathrm{i}} \cdot f(y)1_{\mathrm{j}} = \delta_{\mathrm{ij}} f(y)1_{\mathrm{i}}$,  $\mapsto y_r$, *and crossings act as zero except*

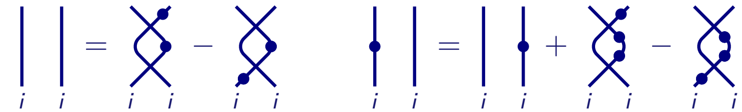
 $\mapsto \begin{cases} \partial_{r,s} & \text{if } i_r = i_s, \\ (r,s) & \text{otherwise} \end{cases}$,  $\mapsto \begin{cases} y_r & \text{if } i_r = i, \\ 1 & \text{otherwise} \end{cases}$,  $\mapsto \begin{cases} Q_{i_r, i_s}(y_r, y_s) & \text{if } i_s \rightsquigarrow i_s, \\ 1 & \text{otherwise} \end{cases}$

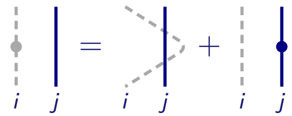
where $\partial_{r,s} = \frac{(r,s)-1}{y_s - y_r}$ is a Demazure operator

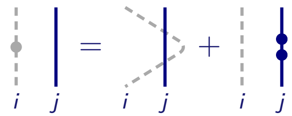
To state these results properly we need to specify the allowed endpoints of the diagrams, the ghost shifts etc. This is analogous to specifying the allowed weights (compositions, partitions, ...), for Schur algebras

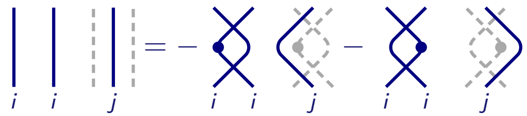
Pulling dots and strings to the right

The KLRW relations allow us to pull strings and dots to the right using identities like:

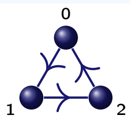


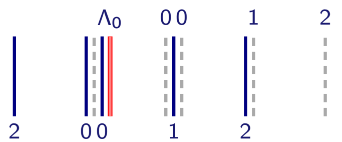
If $i \to j$ then 

If $i \Rightarrow j$ then 

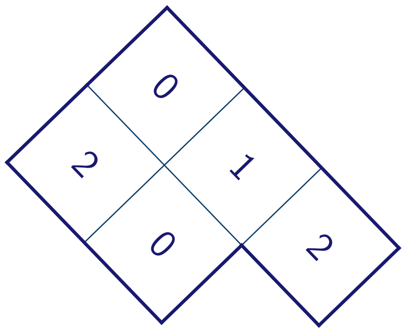


Steady and unsteady strings in type A

Imagine sliding strings in from left to right for $A_2^{(1)} =$ 

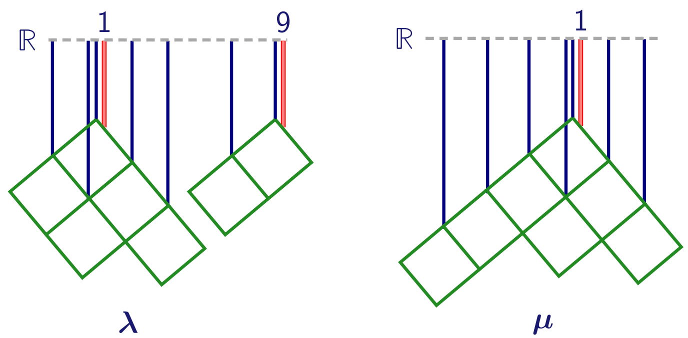


These string are blocked $\Longrightarrow$ steady



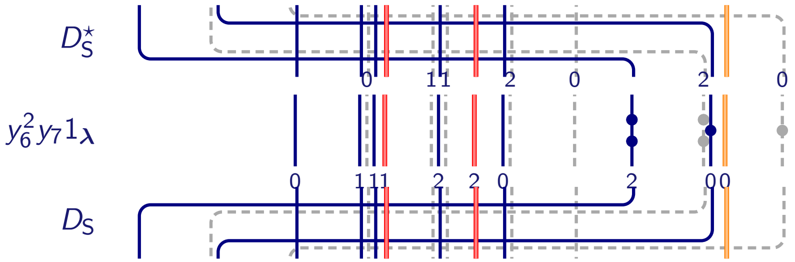
Idempotent diagrams in type A

Following Webster and Bowman, in type $A_{e-1}^{(1)}$ we can write idempotent diagrams for each ℓ-partition:



Diagrammatic basis elements

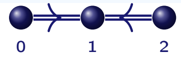
Adding permutations at the top and bottom gives basis elements:

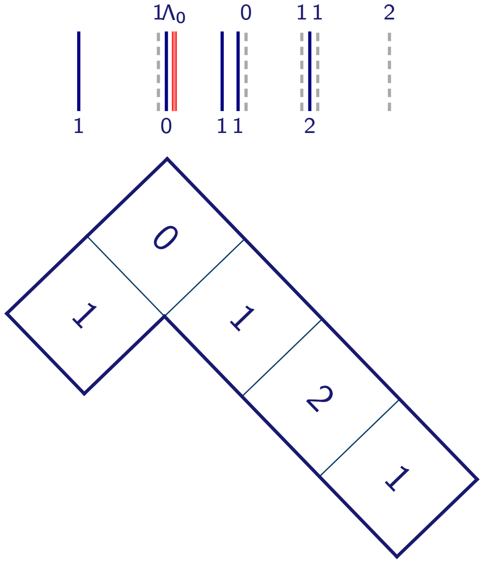


For each pair (S, T) of λ-tableaux of the same shape, and each tuple $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, we obtain bases elements $D^a_{ST} = D^*_S y^a y_\lambda 1_\lambda D_T$

Steady and unsteady strings in type C

Imagine sliding strings in from left to right for $C_2^{(1)}$





Cellular basis combinatorics in affine types

By pulling strings to the right we can associate "dotted idempotents" $y_\lambda 1_\lambda$ for each "ℓ-partition" λ in the following types:

Type	Combinatorics	Residue pattern ($e = 3$)
$A_{e-1}^{(1)}$	partitions	0 1 2 0 1 2 0 1 ⋯
$C_{e-1}^{(1)}$	partitions	0 1 2 1 0 1 2 1 ⋯
$A_{2e-2}^{(2)}$	partitions	0 1 2 2 1 0 1 2 ⋯
	strict partitions	0 1 2 2 1 0 1 2 ⋯
$D_{e-1}^{(2)}$	partitions	0 1 2 2 1 0 0 1 ⋯
	strict partitions	0 1 2 2 1 0 0 1 ⋯

Strict partitions appear for a red string Λ_i when i is a multisink: $? \Longrightarrow i$

$\rightsquigarrow$ For each pair (S, T) of λ-tableaux of the same shape, and each tuple $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, we obtain basis elements

$$D^a_{ST} = D^*_S y^a 1_\lambda D_T$$

such that $a_k \neq 0$ only for strings around affine strings (♠)

Cellular basis combinatorics in affine types

Theorem (M.–Tubbenhauer)

Suppose that Γ is a quiver of type $A_{e-1}^{(1)}$, $C_{e-1}^{(1)}$, $A_{2e-2}^{(2)}$ or $D_{e-1}^{(2)}$. Then W_n is an affine cellular algebra

- Affine cellular means that the layers of the cell filtration of W_n are of the form $C^{\lambda *} \otimes k[y_1, \ldots, y_n]/I \otimes C^\lambda$
- We add $n|I|$ affine strings to catch the strings that are pulled past the red strings. These strings can have arbitrarily many dots, so their cell modules are tensored with the full polynomial ring $k[y_1, \ldots, y_n]$
- Polynomial rings $k[y]/(y^2)$ are attached to cell modules when they have adjacent strings of the same residue in 1_λ
- The proof works by showing that every diagram factors through a (dotted) idempotent diagram $y_\lambda 1_\lambda$ by pulling strings to the right
- The polynomial representation of W_n is used to prove that the basis elements are linearly independent

Cyclotomic KLRW algebras

The cyclotomic KLR algebra W_n^Λ is the quotient of W_n by the ideal generated by the unsteady diagrams

Theorem (M.–Tubbenhauer)

Suppose that Γ is a quiver of type $A^{(1)}_{e-1}$, $C^{(1)}_{e-1}$, $A^{(2)}_{2e-2}$ or $D^{(2)}_{e-1}$ Then W_n^Λ is an affine cellular algebra.

- In type $A^{(1)}_{e-1}$ this is due to Webster and Bowman
- The key point is that we have a basis for the ideal generated by the unsteady diagrams
- This gives cellular bases for the cyclotomic KLR algebras $\mathscr{R}_n^\Lambda$
- The algebra W_n^Λ is quasi-hereditary in type $A^{(1)}_{e-1}$. This is not usually true for other types
- We expect that we will soon be able to complete the classification of the (affine) cellular cyclotomic KLR(W) algebras
- Using W_n^Λ in type $A^{(1)}_{e-1}$, Hu-M.-Rostam proved that the cyclotomic KLR algebras of type $G(r,p,n)$ are skew cellular algebras

Seminormal forms for KLR algebras

Proposition (Evseev-M.)

The cyclotomic KLR algebra $R_n^\Lambda(\mathbb{K}[x^\pm])$ has a unique irreducible (graded) representation $V_{\boldsymbol{\lambda}}$ with basis $\{\, v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$ such that

$$1_{\mathrm{i}} v_{\mathfrak{t}} = \delta_{\mathrm{i}\,\mathrm{r}(\mathfrak{t})} v_{\mathfrak{t}}, \quad y_k v_{\mathfrak{t}} = c_k(\mathfrak{t}) v_{\mathfrak{t}} \quad \text{and} \quad \psi_k v_{\mathfrak{t}} = \beta_k(\mathfrak{t}) v_{\mathfrak{s}} + \frac{1}{c_k(\mathfrak{t}) - c_{k+1}(\mathfrak{t})} v_{\mathfrak{t}}$$

where $\mathfrak{s} = (k, k+1)\mathfrak{t}$ and $\beta_k(\mathfrak{t}) \in \mathbb{K}[x^\pm]$ are prescribed scalars

- The proof is by checking the KLR relations
- Gives all irreducible $R_n^\Lambda(\mathbb{K}[x^\pm])$-modules, for $\boldsymbol{\lambda}$ an ℓ-partition of n
- This implies that $R_n^\Lambda(\mathbb{K}[x^\pm])$ is a direct sum of matrix algebras
- If $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $F_{\mathfrak{t}} = \prod_k \prod_{\mathfrak{s}} \frac{y_k - c_k(\mathfrak{s})}{c_k(\mathfrak{t}) - c_k(\mathfrak{s})} \in R_n^\Lambda(\mathbb{K}[x^\pm])$ is a primitive idempotent
- The degree of the homogeneous basis elements $v_{\mathfrak{t}}$ is not very meaningful because multiplication by x^k arbitrarily shifts the grading
- The module $V_{\boldsymbol{\lambda}}$ is not irreducible as an $R_n^\Lambda(\mathbb{k}[x])$-module
- These modules have very explicit descriptions in types $A^{(1)}_{e-1}$ and $C^{(1)}_{e-1}$

Deformed KLR algebras

For the remainder of the talk assume that Γ is of type $A^{(1)}_{e-1}$ or $C^{(1)}_{e-1}$

This is joint work with Anton Evseev

The presentation of the cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ depends on Rouquier's Q-polynomials $Q_{i,j}(u,v)$

By deforming the Q-polynomials we obtain a deformation of $\mathscr{R}_n^\Lambda$

A standard choice for the Q-polynomials is:

$$Q_{i,j}(u,v) = \begin{cases} u - v & \text{if } i \to j \\ (u-v)(v-u) & \text{if } i \leftrightarrows j \\ u - v^2 & \text{if } i \Rightarrow j \end{cases} \quad \rightsquigarrow \quad \mathscr{R}_n^\Lambda$$

$\Longrightarrow$ We deform the KLR algebra by deforming the Q-polynomials:

$$Q^x_{i,j}(u,v) = \begin{cases} u + x - v & \text{if } i \to j \\ (u+x-v)(v-u-x) & \text{if } i \leftrightarrows j \\ u - (v-x)^2 & \text{if } i \Rightarrow j \end{cases} \quad \rightsquigarrow \quad R_n^\Lambda$$

We actually allow more general deformations

Bases for the deformed cyclotomic KLR algebras

Inspired by symmetric group combinatorics, and the dominance and reverse dominance orders on ℓ-partitions, we can define several bases for R_n^Λ

Let $\mathfrak{s}$ and $\mathfrak{t}$ be standard $\boldsymbol{\lambda}$-tableaux and define:

- $\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}} = \psi^*_{d^\triangleleft_{\mathfrak{s}}} y^\triangleleft_{\boldsymbol{\lambda}} 1_{\mathrm{i}^\triangleleft_{\boldsymbol{\lambda}}} \psi_{d^\triangleleft_{\mathfrak{t}}}$ — reverse dominance order $\triangleleft$
- $f^\triangleleft_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} \psi^\triangleleft_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$ — reverse dominance order $\triangleleft$
- $\psi^\triangleright_{\mathfrak{s}\mathfrak{t}} = \psi^*_{d^\triangleright_{\mathfrak{s}}} y^\triangleright_{\boldsymbol{\lambda}} 1_{\mathrm{i}^\triangleright_{\boldsymbol{\lambda}}} \psi_{d^\triangleright_{\mathfrak{t}}}$ — dominance order $\triangleright$
- $f^\triangleright_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} \psi^\triangleright_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}$ — dominance order $\triangleright$

By definition, $\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}}, \psi^\triangleright_{\mathfrak{s}\mathfrak{t}} \in R_n^\Lambda(\mathbb{k}[x])$ and $f^\triangleleft_{\mathfrak{s}\mathfrak{t}}, f^\triangleright_{\mathfrak{s}\mathfrak{t}} \in R_n^\Lambda(\mathbb{K}[x^\pm])$

Specialising $x = 0$, it is immediate that $\psi^\triangleleft_{\mathfrak{s}\mathfrak{t}}, \psi^\triangleright_{\mathfrak{s}\mathfrak{t}} \in \mathscr{R}_n^\Lambda(\mathbb{k})$

In type $A^{(1)}_{e-1}$, the two ψ-bases generalise bases for $\mathscr{R}_n^\Lambda$ that I previously constructed with Hu. We were only able to understand these bases using the Brundan–Kleshchev graded isomorphism theorem

In type $C^{(1)}_{e-1}$, these bases are completely new

Cellularity over $\mathbb{K}$

Theorem

Suppose that (c,r) *is a content system. Then the algebra* $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$ *is a graded* $\mathbb{K}[x^\pm]$*-cellular algebra with cellular bases:*

- $\{ f_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ f_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$

Moreover, if $\boldsymbol{\lambda} \in \mathcal{P}_{\ell,n}$ *and* $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_{\ell,n})$ *then*

$$V_{\boldsymbol{\lambda}} \cong \mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm]) f_{\mathfrak{st}}^{\triangleleft} = \mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm]) f_{\mathfrak{st}}^{\triangleright} = \mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm]) \psi_{\mathfrak{st}}^{\triangleleft} = \mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm]) \psi_{\mathfrak{st}}^{\triangleright}$$

This recovers almost everything from the semisimple representation theory of the symmetric group in the representation theory of $\mathsf{R}_n^\Lambda(\mathbb{K}[x^\pm])$.

Although this is cute, what we really want are cellular bases for the non-semisimple algebras $\mathsf{R}_n^\Lambda(\Bbbk[x])$, which will give cellular bases for $\mathscr{R}_n^\Lambda(\Bbbk)$

Cyclotomic categorification

Recall that $U_q(\mathfrak{g}_\Gamma)$ is the Kac-Moody algebra associated with the quiver Γ and let $U_{\mathcal{A}}$ be its integral form, where $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and that $L(\Lambda)$ is an irreducible integrable highest weight module for $U_q(\mathfrak{g}_\Gamma)$

Let $\mathrm{Rep}\,\mathsf{R}_n^\Lambda(\Bbbk[x])$ be the category of finite dimensional graded $\mathsf{R}_n^\Lambda(\Bbbk[x])$-modules and let $\mathrm{Proj}\,\mathsf{R}_n^\Lambda(\Bbbk[x])$ be the subcategory of projective modules.

Using the bases $\{\psi_{\mathfrak{st}}^{\triangleleft}\}$ and $\{f_{\mathfrak{st}}^{\triangleleft}\}$, and $\{\psi_{\mathfrak{st}}^{\triangleright}\}$ and $\{f_{\mathfrak{st}}^{\triangleright}\}$, it is easy to prove graded branching rules for the graded Specht modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk[x])$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk[x])$

$\rightsquigarrow$ Up to shift, $U_{\mathcal{A}}$ acts on the Grothendieck groups of $\mathsf{R}_n^\Lambda(\Bbbk[x])$ via the natural i-induction and i-restriction functors

Theorem (Cyclotomic categorification)

As $U_{\mathcal{A}}$*-modules,* $L(\Lambda)_{\mathcal{A}} \cong \bigoplus_{n\ge 0} [\mathrm{Proj}\,\mathsf{R}_n^\Lambda(\Bbbk[x])]$

and $L(\Lambda)_{\mathcal{A}}^* \cong \bigoplus_{n\ge 0} [\mathrm{Rep}\,\mathsf{R}_n^\Lambda(\Bbbk[x])]$

There are two variations of this result, for the $\psi^{\triangleleft}$ and $\psi^{\triangleright}$ bases

Integral cellularity

Theorem (Evseev-M.)

Suppose that (c,r) *is a content system. Then the algebra* $\mathsf{R}_n^\Lambda(\Bbbk[x])$ *is a graded* $\Bbbk[x]$*-cellular algebra with graded cellular bases:*

- $\{ \psi_{\mathfrak{st}}^{\triangleleft} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglelefteq)$
- $\{ \psi_{\mathfrak{st}}^{\triangleright} \mid (\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_{\ell,n}) \}$ *with weight poset* $(\mathcal{P}_{\ell,n}, \trianglerighteq)$

The trickiest part of the proof is showing that these bases span the algebra over $\Bbbk[x]$. The key idea is to argue by induction on the cell poset $\mathcal{P}_{\ell,n}$

Corollary

For any domain $\Bbbk$*, the cyclotomic KLR algebras* $\mathscr{R}_n^\Lambda(\Bbbk)$ *of types* $A_{e-1}^{(1)}$ *and* $C_{e-1}^{(1)}$ *are graded cellular algebras with graded cellular bases* $\{\psi_{\mathfrak{st}}^{\triangleleft}\}$ *and* $\{\psi_{\mathfrak{st}}^{\triangleright}\}$

The cell modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk[x])$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk[x])$ defined by the $\psi^{\triangleleft}$ and $\psi^{\triangleright}$ bases are analogues of the graded Specht modules for $\mathsf{R}_n^\Lambda(\Bbbk[x])$

$\rightsquigarrow$ Gives graded Specht modules $S_{\boldsymbol{\lambda}}^{\triangleleft}(\Bbbk)$ and $S_{\boldsymbol{\lambda}}^{\triangleright}(\Bbbk)$ for $\mathscr{R}_n^\Lambda(\Bbbk)$

Simple modules for R_n^Λ and $\mathscr{R}_n^\Lambda$

Cellular machinery gives $D_{\boldsymbol{\mu}}^{\triangleleft} = S_{\boldsymbol{\mu}}^{\triangleleft} / \operatorname{Rad} S_{\boldsymbol{\mu}}^{\triangleleft}$ and $D_{\boldsymbol{\mu}}^{\triangleright} = S_{\boldsymbol{\mu}}^{\triangleright} / \operatorname{Rad} S_{\boldsymbol{\mu}}^{\triangleright}$

Let $Y_{\boldsymbol{\mu}}^{\triangleleft}$ and $Y_{\boldsymbol{\mu}}^{\triangleright}$ be the projective covers of these modules

$\Longrightarrow$ $\{[D_{\boldsymbol{\mu}}^{\triangleleft}]\}$ and $\{[D_{\boldsymbol{\mu}}^{\triangleright}]\}$ are bases for $L(\Lambda)_{\mathcal{A}}^*$, and $\{[Y_{\boldsymbol{\mu}}^{\triangleleft}]\}$ and $\{[Y_{\boldsymbol{\mu}}^{\triangleright}]\}$ are bases for $L(\Lambda)_{\mathcal{A}}$

$\Longrightarrow$ There are bar-invariant transition matrices between these bases of the global/canonical bases of $L(\Lambda)_{\mathcal{A}}^*$ and $L(\Lambda)_{\mathcal{A}}$

$\Longrightarrow$ Using good node sequences, or paths in the crystal graph, we can define sets $\mathcal{K}_n^{\triangleleft}$ and $\mathcal{K}_n^{\triangleright}$ such that:

Theorem (Evseev-M.)

1. $\{ q^d D_{\boldsymbol{\mu}}^{\triangleleft} \mid d \in \mathbb{Z}, \boldsymbol{\mu} \in \mathcal{K}_n^{\triangleleft} \}$ *is a complete set of simple* R_n^Λ*-modules*
2. $\{ q^d D_{\boldsymbol{\mu}}^{\triangleright} \mid d \in \mathbb{Z}, \boldsymbol{\mu} \in \mathcal{K}_n^{\triangleright} \}$ *is a complete set of simple* R_n^Λ*-modules*
3. *There is an explicit Mullineux map* $\mathsf{m} : \mathcal{K}_n^{\triangleleft} \longrightarrow \mathcal{K}_n^{\triangleright}$ *such that* $D_{\mathsf{m}(\boldsymbol{\mu})}^{\triangleright} \cong D_{\boldsymbol{\mu}}^{\triangleleft}$*. Moreover,* $(D_{\boldsymbol{\mu}}^{\triangleleft})^{\mathrm{sgn}} \cong D_{\mathsf{m}(\boldsymbol{\mu})'}^{\triangleleft}$ *and* $(D_{\boldsymbol{\mu}}^{\triangleright})^{\mathrm{sgn}} \cong D_{\mathsf{m}^{-1}(\boldsymbol{\mu})'}^{\triangleright}$
4. *We have modular branching rules for* $D_{\boldsymbol{\mu}}^{\triangleleft}$ *and* $D_{\boldsymbol{\mu}}^{\triangleright}$

Canonical bases and decomposition matrices

Graded decomposition numbers: $d^{\triangleleft}_{\boldsymbol{\lambda\mu}}(q) = \sum_{d\in\mathbb{Z}} [S^{\triangleleft}_{\boldsymbol{\lambda}} : q^d D^{\triangleleft}_{\boldsymbol{\mu}}]_q\, q^d \in \mathbb{N}[q, q^{-1}]$

In type $A^{(1)}_{e-1}$, (Ariki and) Brundan and Kleshchev proved that the images of the simple $\mathscr{R}^{\Lambda}_n(\mathbb{C})$-modules in $L(\Lambda)^*_{\mathcal{A}}$ coincide with the dual canonical basis
$\implies \quad d^{\triangleleft}_{\boldsymbol{\lambda\mu}}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{N}[q]$, a parabolic Kazhdan-Lusztig polynomial

It is natural to ask if this is still true in type $C^{(1)}_{e-1}$

In general, in type $C^{(1)}_{e-1}$ the dual canonical basis does not coincide with basis of simple modules, as shown by the decomposition matrix for the principal block of $\mathscr{R}^{\Lambda_0}_8(\mathbb{Q})$ for the quiver $C^{(1)}_2$

(Chung-M.-Speyer)

	8	$7,1$	$6,1^2$	$5,1^3$	4^2	$4,3,1$
8	1					
$7,1$	q	1				
$6,1^2$	q	q^2	1			
$5,1^3$	q^2	.	q	1		
4^2	.	q	.	.	1	
$4,3,1$	q^2+1	q^3	q	.	q^2	1
$4,2^2$	$2q$	.	q^2	.	.	q
$4,2,1^2$	$2q^2$	.	q^3+q	q^2	.	q^2
$4,1^4$	q^2	.	q^3	q^4	.	.
$3^2,2$	$2q^2$	.	.	.	.	q^2
$3^2,1^2$	$2q^3$	.	q^2	.	.	q^3
$3,2^2,1$	q^4+q^2	q	q^3	.	q^2	q^4
$3,1^5$	q^3	q^2	q^4	.	.	.
2^4	.	q^3	.	.	q^4	.
$2,1^6$	q^3	q^4	.	.	.	.
1^8	q^4	.	.	.	.	.

Non-polynomial decomposition number – Chung-M.-Speyer

Type $C^{(1)}_2$ over $\mathbb{C}$

	1^{13}	$2^2,1^9$	$2^4,1^5$	$2^6,1$	$3,2,1^8$	$3,2^2,1^6$	$3,2^5$	$3^3,1^4$	$3^3,2,1^2$	$4,2,1^7$	$4,3^2,2,1$	$5,2,1^6$	$5,2^2,1^4$
1^{13}	1												
$2^2,1^9$	q	1											
$2^4,1^5$	.	q	1										
$2^6,1$	.	.	q	1									
$3,2,1^8$	q	q^2	.	.	1								
$3,2^2,1^6$	$1+q^2$	q^3	q^2	.	q	1							
$3,2^5$	.	.	q^3	q^2	.	q	1						
$3^2,1^7$	$2q$	.	.	.	q^2	q	.						
$3^3,1^4$	$2q^2$	q	q^2	q	.	q^2	.	1					
$3^3,2,1^2$	$1+q^2$	q^3	q^4	q^3	q	q^2	q	q^2	1				
$3^3,2^2$	$q+q^3$	.	.	.	q^2	q^3	q^2	.	q				
$3^4,1$	$2q^2$	.	.	.	q^3	.	.	.	q^2				
$4,2,1^7$	$2q^2$	.	.	.	q^3	q^2	.	.	.	1			
$4,3^2,2,1$	$q^{-1}+q+3q^3$	q^2	.	.	q^4	q^3	q^2	q	q^3	q	1		
$4^2,3,1^2$	$q+2q^3$	q^4	.	.	.	.	.	q^3	.	q	q^2		
$5,1^8$	q^2	.	.	.	.	.	.	.	.	q^2	.		
$5,2,1^6$	$2q^2$	.	.	.	q	q^2	.	.	.	q^4	.	1	
$5,2^2,1^4$	$2q^3$	q^2	q^3	q^2	q^2	q^3	.	q	.	.	.	q	1
$5,2^3,1^2$	$q+q^3$	q^4	q^5	q^4	q^2	q^3	q^2	q^3	q	.	.	q	q^2
$5,2^4$	q^2+q^4	.	.	.	q^3	q^4	q^3	.	q^2	.	.	q^2	.
$5,3,1^5$	q^2+q^4	q	.	.	q^3	q^4	.	q^2	.	.	.	q^2	q
$5,3,2^2,1$	$q+3q^3+q^5$	q^2	.	q^2	q^4	q^5	q^4	$q+q^3$	q^3	q^3	q^2	q^3	q^2
$5,3^2,2$	$2q^3+q^5$	q^2	.	.	.	.	.	q^3	.	q^5	q^4	.	.
$5,4,1^4$	$2q^2$	q^3	.	.	q	.	.	q^4	q^2	.	.	q^2	q^3
$5,4,2,1^2$	$q+3q^3+q^5$	q^4	.	q^4	q^2	q	q^2	q^3+q^5	q^3	q^3	q^4	q^3	q^4
$5,4,3,1$	$3q^3+q^5+q^7$	q^4	.	.	q^2	q^3	q^4	q^5	q^3	q^5	q^6	.	.

A negative canonical basis coefficient (Chung-M.Speyer)

Type $C^{(1)}_2$ over $\mathbb{C}$

	1^{12}	$2,1^{10}$	$2^4,1^4$	$2^5,1^2$	$3,1^9$	$3,2^2,1^5$	$3,2^4,1$	$3^2,2,1^4$	$3^2,2^2,1^2$	$4,1^8$	$4,2,1^6$	$4,2^2,1^4$	$4,3,2^2,1$
1^{12}	1												
$2,1^{10}$	q	1											
$2^4,1^4$	.	q	1										
$2^5,1^2$	.	.	q	1									
$3,1^9$	q	q^2	.	.	1								
$3,2^2,1^5$	$1+q^2$	q^3	q^2	.	q	1							
$3,2^4,1$	.	.	q^3	q^2	.	q	1						
$3^2,1^6$	$2q$	.	.	.	q^2	q	.						
$3^2,2,1^4$	$2q^2$	q	q^2	q	.	q^2	.	1					
$3^2,2^2,1^2$	$1+q^2$	q^3	q^4	q^3	q	q^2	q	q^2	1				
$3^2,2^3$	$q+q^3$	.	.	.	q^2	q^3	q^2	.	q				
3^4	$2q^2$	.	.	.	q^3	.	.	.	q^2				
$4,1^8$	q^2	.	.	.	q	.	.	.	.	1			
$4,2,1^6$	$2q^2$	.	.	.	$q+q^3$	q^2	.	.	.	q^2	1		
$4,2^2,1^4$	$2q^3$	q^2	q^3	q^2	q^2	q^3	.	q	.	.	q	1	
$4,2^3,1^2$	$q+q^3$	q^4	q^5	q^4	q^2	q^3	q^2	q^3	q	.	q	q^2	
$4,2^4$	q^2+q^4	.	.	.	q^3	q^4	q^3	.	q^2	.	q^2	.	
$4,3,1^5$	q^2+q^4	q	.	.	q^3	q^4	.	q^2	.	.	q^2	q	
$4,3,2^2,1$	$q+2q^3+q^5$	$1+q^2$	.	q^2	q^2+q^4	q^5	q^4	q^3	$q+q^3$	q	q^3	q^2	1
$4,3^2,2$	$2q^3$	$2q^2$	.	.	q^4	.	.	.	q^3	q^3	.	.	q^2
$4^2,1^4$	$2q^2$	q^3	.	.	q	.	.	q^4	q^2	.	q^2	q^3	.
$4^2,2,1^2$	$2q^3$	q^2+q^4	.	q^4	q^2	.	.	q^5	q^3	q	q^3	q^4	q^2
$4^2,3,1$	$2q^3$	$2q^4$	.	.	q^2	.	.	.	q^3	q^3	.	.	q^4
4^3	$2q^4$	.	.	.	q^3	.	.	.	q^4	.	.	.	.
$5,1^7$	q^2	.	.	.	q^3	.	.	.	.	q^4	q^2	.	.
$5,3,2^2$	$2q^3$	q^2+q^4	.	q^2	q^4	.	.	q	q^3	q^5	q^3	q^2	q^4
$5,4,2,1$	$q+2q^3+q^5$	q^4+q^6	.	q^4	q^2+q^4	q	q^2	q^3	q^3+q^5	q^5	q^3	q^4	q^6
$5^2,1^2$	q^2+q^4	.	.	.	q^3	q^2	q^3	.	q^4	.	q^4	.	.
$5^2,2$	q^3+q^5	.	.	.	q^4	q^3	q^4	.	q^5	.	.	.	.
$6,1^6$	q^3	q^2	.	.	q^4	.	.	.	.	.	q^3	q^2	.
$6,2^3$	$2q^4$	q^3	.	.	q^5	.	.	q^2	q^4	.	q^4	q^3	.
$6,4,1^2$	q^3+q^5	q^2	q	q^2	q^4	q^3	q^4	q^3	q^5	.	q^5	q^4	.
$6,4,2$	q^4+q^6	q^3	q^2	q^3	q^5	q^4	q^5	q^4	q^6	.	.	.	.
$6,5,1$	.	.	q^3	q^4	.	q^5	q^6	.	.	.	.	.	.
$7,1^5$	q^3	q^4	.	.	q^2	.	.	.	.	.	q^3	q^4	.
$7,2^2,1$	q^2+q^4	q^5	.	.	q^3	q^2	.	q^4	.	.	q^4	q^5	.
$7,3,1^2$	$2q^3$	q^4	q^3	q^4	q^4	q^3	.	q^5	.	.	q^5	q^6	.
$7,3,2$	$2q^4$	q^5	q^4	q^5	.	q^4	.	q^6	.	.	.	.	.
$7,5$	.	.	q^5	q^6	.	.	.	.	.	.	.	.	.
$8,1^4$	q^4	.	.	.	q^3	.	.	.	.	q^2	q^4	.	.
$8,2,1^2$	$2q^4$	.	.	.	q^3+q^5	q^4	.	.	.	q^4	q^6	.	.
$8,2^2$	$2q^5$	.	.	.	q^4	q^5	.	.	.	.	.	.	.
$8,3,1$	q^4+q^6	q^3	q^4	.	q^5	q^6	.	.	.	.	.	.	.
$8,4$	.	q^5	q^6	.	.	.	.	.	.	.	.	.	.
$9,1^3$	q^4	.	.	.	q^5	.	.	.	.	q^6	.	.	.
$10,1^2$	q^5	q^4	.	.	q^6	.	.	.	.	.	.	.	.
$11,1$	q^5	q^6	.	.	.	.	.	.	.	.	.	.	.
12	q^6	.	.	.	.	.	.	.	.	.	.	.	.