Diagonal cuts for diagrammatic Cherednik algebras

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Background

- The KLR algebra $\mathcal{R}_n$ is a $\mathbb{Z}$-graded algebra, introduced by Khovanov and Lauda [KL], and by Rouquier [R], to categorify quantum groups.
- In type A, the (level l) cyclotomic quotients $\mathcal{R}_n^\Lambda$ of $\mathcal{R}_n$ are isomorphic to the corresponding (level l) cyclotomic Hecke algebras of type A, via a result of Brundan and Kleshchev [BK] – in particular, this allows us to study the graded representation theory of cyclotomic Hecke algebras.
- Just as the classical Schur algebra is a useful tool for studying the representation theory of the symmetric group, we would like a graded quasi-hereditary cover of $\mathcal{R}_n^\Lambda$ to assist in studying its representation theory.
- Such covers have been constructed by Webster. For any weighting $\theta \in \mathbb{Z}^l$, Webster constructed a diagrammatic Cherednik algebra $A(n,\theta,\Lambda)$, a graded quasi-hereditary cover of $\mathcal{R}_n^\Lambda$.
- $A(n,\theta,\Lambda)$ is a graded cellular algebra, with cell modules $\Delta_\theta(\lambda)$ indexed by the set $\mathscr{P}_n^l$ of l-multipartitions of n. The cell modules have simple heads $L_\theta(\lambda)$. Varying θ and applying a truncation functor leads to different graded cellular structures on $\mathcal{R}_n^\Lambda$, each with corresponding cell modules $S_\theta(\lambda)$ for $\lambda \in \mathscr{P}_n^l$ and an indexing set $\Theta \subset \mathscr{P}_n^l$ for the simple $\mathcal{R}_n^\Lambda$-modules $D_\theta(\lambda)$.

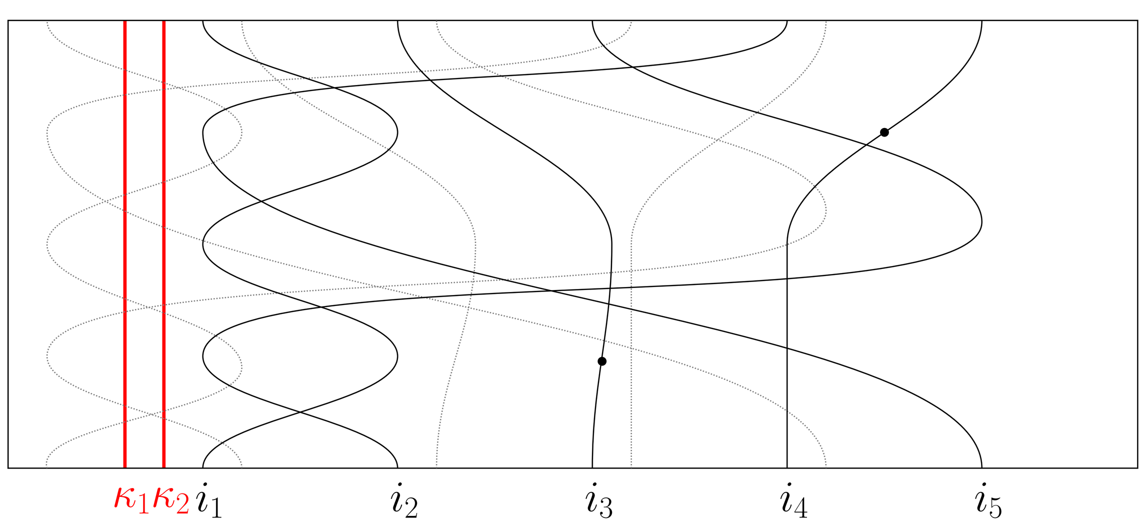
Decomposition Number Problem

For $A(n,\theta,\Lambda)$, we would like to find the composition multiplicities $d_{\lambda\mu} = [\Delta_\theta(\lambda) : L_\theta(\mu)]$ for all $\lambda, \mu \in \mathscr{P}_n^l$.
For $\mathcal{R}_n^\Lambda$, we would like to find the composition multiplicities $[S_\theta(\lambda) : D_\theta(\mu)]$ for all $\lambda \in \mathscr{P}_n^l$, $\mu \in \Theta$.

$$[\Delta_\theta(\lambda) : L_\theta(\mu)] = [S_\theta(\lambda) : D_\theta(\mu)] \text{ if } \mu \in \Theta.$$

The algebra $A(n,\theta,\Lambda)$

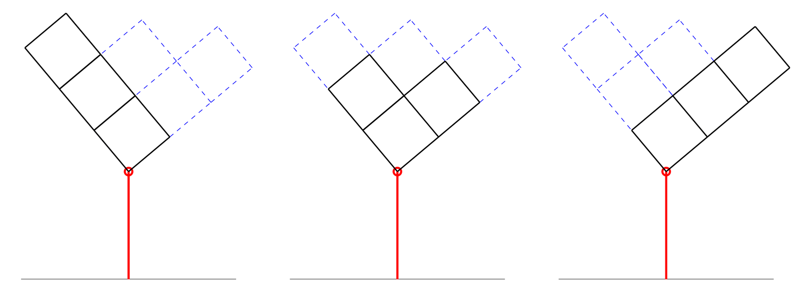
Webster's diagrammatic Cherednik algebra $A(n,\theta,\Lambda)$ is a diagram algebra comprising l vertical red strands, and n black strands which run from top-to-bottom and may carry dots, each possessing a *ghost* to its left. Strands have residues modulo e. Such diagrams look like:



$A(n,\theta,\kappa)$ has a graded cellular basis indexed by a generalisation of semistandard tableaux. A key part of this is the combinatorics of θ-Young diagrams for multipartitions $\lambda \in \mathscr{P}_n^l$.

θ-Young diagrams

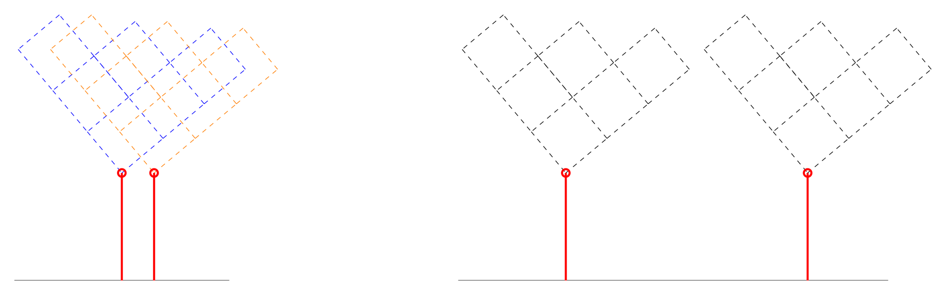
For a partition λ, we draw its Young diagram $[\lambda]$ under a (slightly tilted) mirrored Russian convention, with each box having a diagonal of length $2l$, e.g.



We have an ordering from left-to-right:

$$(3) \trianglerighteq_\theta (2,1) \trianglerighteq_\theta (1^3).$$

For a multipartition λ, we draw its Young diagram by placing the Young diagram for the mth component at x-coordinate θ_m.
We depict the $\theta = (0,1)$ and $\theta = (0,16)$ cases:



These yield different orders on multipartitions, where boxes are weighted depending on how far to the left they appear in the Young diagram.

Diagonal cuts for multipartitions

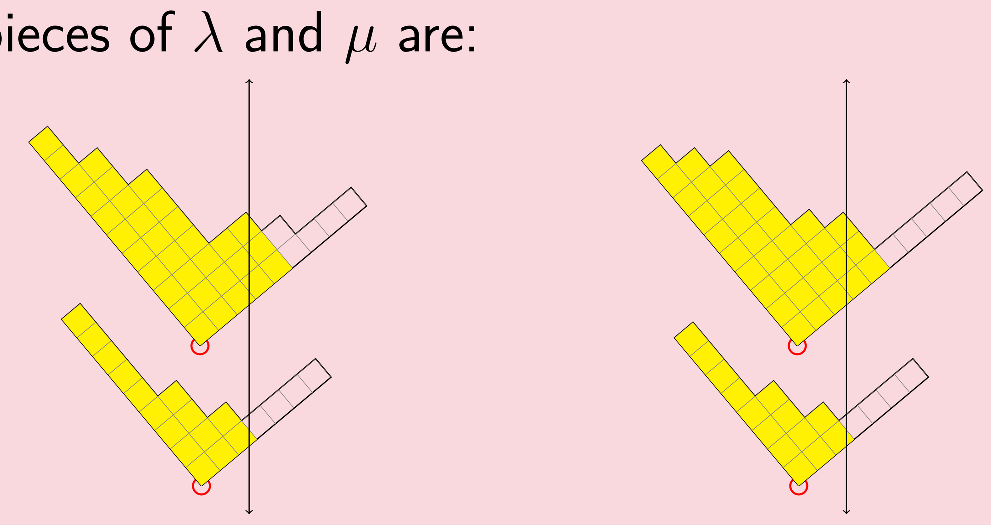
Aim

Produce a reduction-type result for graded decomposition numbers $[\Delta_\theta(\lambda) : L_\theta(\mu)]$.

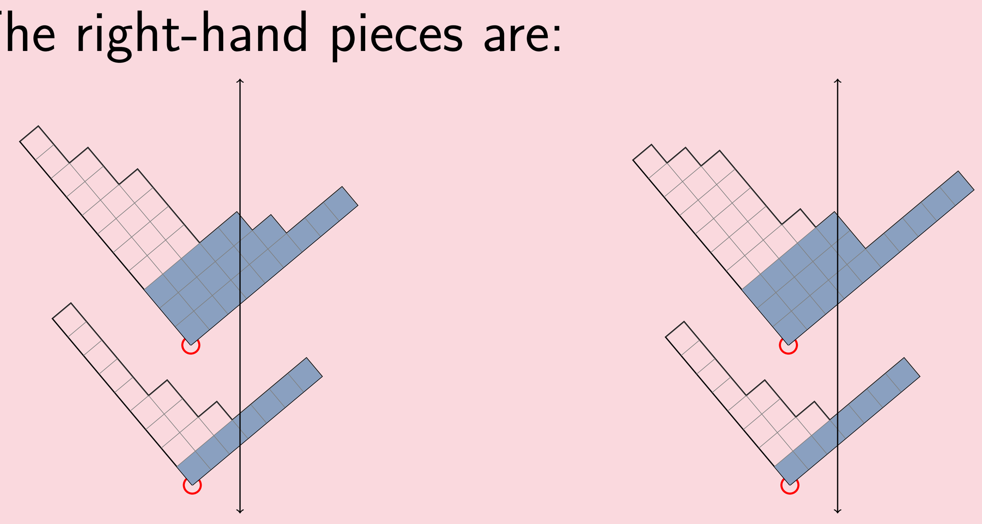
A pair of multipartitions λ, μ admits a diagonal cut at $x = a$ if when we draw the line $x = a$ on $[\lambda]$ and $[\mu]$ we have the same number of boxes to the left of the line in $[\lambda]$ as in $[\mu]$, and likewise to the right of the line. We denote by λ^L and μ^L the minimal multipartitions including all boxes to the left of the line in λ and μ, and by λ^R and μ^R the analogous multipartitions on the right-hand side.

Example

Let $\theta = (0,1)$, and take the bipartitions $\lambda = ((11,9,7,3^2,2,1^3),(9,4,2,1^4))$ and $\mu = ((10,9,8,4,3,1^5),(8,4,2,1^4))$. Then λ, μ admit a diagonal cut at $x = 5.5$. The left-hand pieces of λ and μ are:



The right-hand pieces are:



Diagonal cuts for $d_{\lambda\mu}$

The main result of [BSa] is the following.

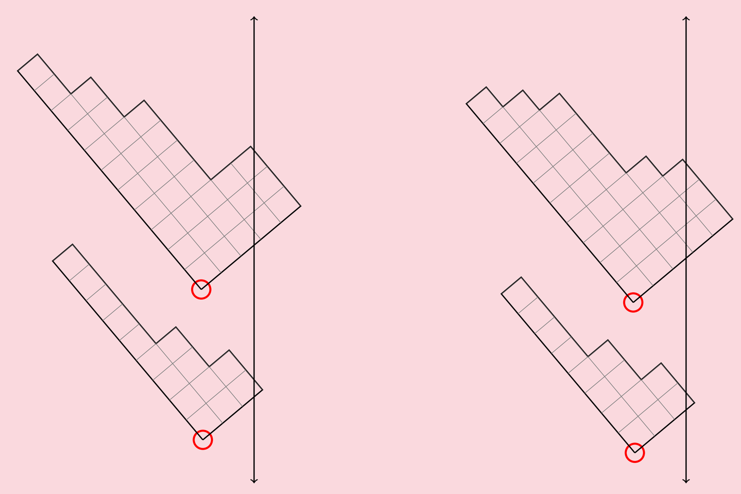
Theorem

Let (λ,μ) be a pair of l-multipartitions of n and let $a \in \mathbb{R}$. If (λ,μ) admits a diagonal cut at $x = a$ into two pieces (λ^L,μ^L) and (λ^R,μ^R), then we can factorise the graded decomposition numbers for these algebras as

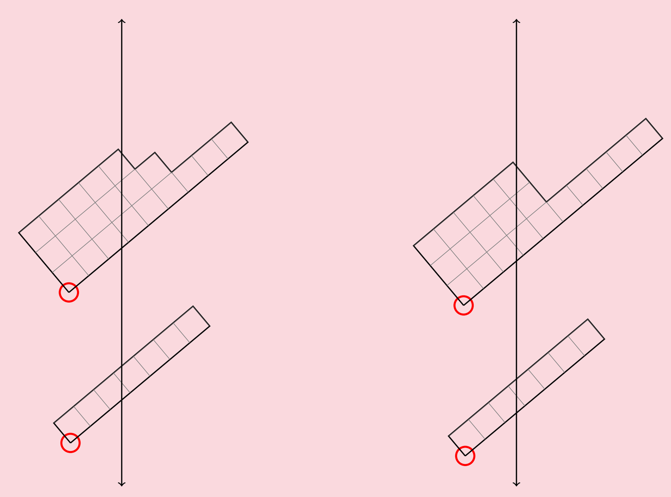
$$d_{\lambda\mu} = d_{\lambda^L\mu^L} \times d_{\lambda^R\mu^R}.$$

Example

Continuing the previous example, (λ^L,μ^L) are:



(λ^R,μ^R) are:



Now let $e = 5$ and $\Lambda = \Lambda_0 + \Lambda_2$. Then by a theorem in [BSb], we may compute that (over a field of any characteristic)

$$d_{\lambda^L\mu^L} = v^5 + v^3 \quad \text{and} \quad d_{\lambda^R\mu^R} = v^2,$$

so that our above theorem yields

$$d_{\lambda\mu} = v^7 + v^5.$$

The theorem used to calculate the graded decomposition numbers is not applicable to $d_{\lambda\mu}$!

References

[BK] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras*, Invent. Math. **178** (2009), 451–484.

[BSa] C. Bowman and L. Speyer, *An analogue of row removal for diagrammatic Cherednik algebras*, arXiv:1601.05543, 2016, preprint.

[BSb] C. Bowman and L. Speyer, *Kleshchev's decomposition numbers for diagrammatic Cherednik algebras*, Trans. Amer. Math. Soc. **370** (2018), no. 5, 3551–3590.

[KL] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.

[R] R. Rouquier, *2-Kac–Moody algebras*, arXiv:0812.5023, (2008).