Rouquier / RoCK blocks for Ariki-Koike algebras

Sinéad Lyle

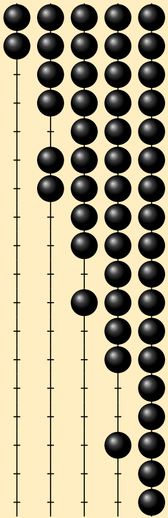
Okinawa Institute of Science and Technology

June 2023

University of East Anglia

Rouquier / RoCK blocks for symmetric groups

Example

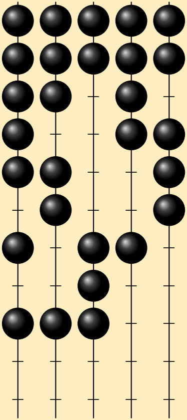


Abacuses and partitions

Definition

Fix $e \geq 2$ and let $\mathscr{A}_e$ denote the set of abacus configurations on e runners. There is a bijection $\Lambda \times \mathbb{Z} \longleftrightarrow \mathscr{A}_e$.

$$((13^3, 11, 8^2, 7^2, 5, 4, 2^4),\ 28) \longleftrightarrow$$



The Nakayama Conjecture

The Nakayama Conjecture

Let $\mathbb{F}$ be a field of characteristic p. Suppose that $\lambda, \mu \in \Lambda_n$. The $\mathbb{F}\mathfrak{S}_n$-modules S^λ and S^μ lie in the same block if and only if

$$\bar{\lambda} = \bar{\mu} \text{ and } \mathrm{w}(\lambda) = \mathrm{w}(\mu)$$

$\Longrightarrow$ A block is determined by its core and weight.

The Nakayama Conjecture

The Nakayama Conjecture

Let $\mathbb{F}$ be a field of characteristic p. Suppose that $\lambda, \mu \in \Lambda_n$. The $\mathbb{F}\mathfrak{S}_n$-modules S^λ and S^μ lie in the same block if and only if

$$\bar{\lambda} = \bar{\mu} \text{ and } \mathrm{w}(\lambda) = \mathrm{w}(\mu)$$

$\Longrightarrow$ A block is determined by its core and weight.

On the Nakayama Conjecture

It seems to the author that the value of this Theorem [the Nakayama Conjecture] has been overrated; it is certainly useful (but not essential) when trying to find the decompositions matrix of $\mathfrak{S}_n$ for a particular small n, but there are few general theorems in which it is helpful.

– Gordon James (1978)

A map between blocks

Definition

Let $0 \leq i \leq e-2$. Define a map $\Phi_i : \Lambda \to \Lambda$ where $\Phi_i(\lambda)$ is obtained by swapping runners i and $i+1$ on the abacus configuration for λ.

Lemma

S^λ and S^μ lie in the same block if and only if $S^{\Phi_i(\lambda)}$ and $S^{\Phi_i(\mu)}$ lie in the same block.

Scopes equivalence

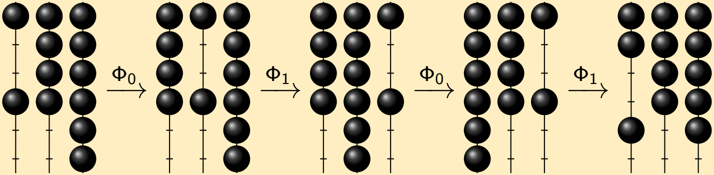
Definition

Let $\lambda \in \Lambda$. For $0 \leq i \leq e-1$, let $\mathfrak{b}_i$ be the number of beads on runner i of the abacus configuration of $\bar{\lambda}$. If $|\mathfrak{b}_{i+1} - \mathfrak{b}_i| \leq \mathrm{w}(\lambda)$, say that Φ_i is a Scopes map.

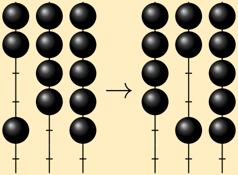
Define an equivalence relation $\sim_{\mathsf{Sc}}$ on the set of blocks of the symmetric group algebras by the closure of the relation $B \sim_{\mathsf{Sc}} B'$ if $B = \Phi_i(B')$ for Φ_i a Scopes map.

Examples of Scopes equivalence

Example



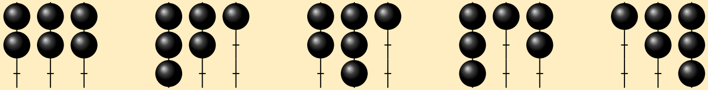
Not allowed:



Scopes equivalence classes

Example

Let $e = 3$. Up to Scopes equivalence, the cores of the blocks of weight 2 are

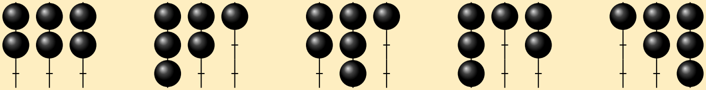


$\emptyset$ (1) (2) (1^2) $(3, 1^2)$

Scopes equivalence classes

Example

Let $e = 3$. Up to Scopes equivalence, the cores of the blocks of weight 2 are



$\emptyset$ (1) (2) (1^2) $(3, 1^2)$

Theorem (Scopes)

Suppose the blocks B and B' are Scopes equivalent. Then they are Morita equivalent and decomposition equivalent.

Rouquier / RoCK blocks

Definition

Let $\lambda \in \Lambda$. Say that λ is a Rouquier partition if $\mathfrak{b}_{i+1} - \mathfrak{b}_i + 1 \geq \mathrm{w}(\lambda)$ for all $0 \leq i \leq e - 2$.

Rouquier / RoCK blocks

Definition

Let $\lambda \in \Lambda$. Say that λ is a Rouquier partition if $\mathfrak{b}_{i+1} - \mathfrak{b}_i + 1 \geq \mathrm{w}(\lambda)$ for all $0 \leq i \leq e-2$.

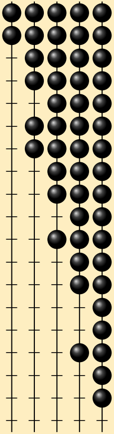
Definition

If S^λ and S^μ lie in the same block then λ is a Rouquier partition if and only if μ is a Rouquier partition. We then say that the block is a Rouquier block.

If a block is Scopes equivalent to a Rouquier block, we call it a RoCK block.

A Rouquier partition

Example



$\mathfrak{b}_0 = 2$
$\mathfrak{b}_1 = 6$
$\mathfrak{b}_2 = 10$
$\mathfrak{b}_3 = 14$
$\mathfrak{b}_4 = 18$

$w(\lambda) = 5$

The Ariki-Koike algebras

Definition

For each $n \geq 0$, $r \geq 1$, $e \geq 2$ and $\boldsymbol{a} \in I^r$ where $I = \{0, 1, \ldots, e-1\}$ we have an Ariki-Koike algebra $\mathcal{H}_{r,n}(\boldsymbol{a})$.

The Ariki-Koike algebras

Definition

For each $n \geq 0$, $r \geq 1$, $e \geq 2$ and $\boldsymbol{a} \in I^r$ where $I = \{0, 1, \ldots, e-1\}$ we have an Ariki-Koike algebra $\mathcal{H}_{r,n}(\boldsymbol{a})$.

Specht modules are indexed by r-multipartitions of n:

$$\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)}).$$

The Ariki-Koike algebras

Definition

For each $n \geq 0$, $r \geq 1$, $e \geq 2$ and $\boldsymbol{a} \in I^r$ where $I = \{0, 1, \ldots, e-1\}$ we have an Ariki-Koike algebra $\mathcal{H}_{r,n}(\boldsymbol{a})$.

Specht modules are indexed by r-multipartitions of n:

$$\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)}).$$

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^r$.

- Say that $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ if $S^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\mu}}$ lie in the same $\mathcal{H}_{r,n}(\boldsymbol{a})$-block.
- Equivalently $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ if $\mathrm{Res}(\boldsymbol{\lambda})$=$\mathrm{Res}(\boldsymbol{\mu})$.
- Say that $\boldsymbol{\lambda} \approx \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ and $\bar{\boldsymbol{\lambda}} = \bar{\boldsymbol{\mu}}$.

Call the $\sim$-equivalence classes blocks.

Note that if $r = 1$ then $\sim \iff \approx$.

Abacus configurations for $\mathcal{H}_{r,n}(\boldsymbol{a})$

Definition

Let $\mathcal{H}_{r,n}(\boldsymbol{a})$ be an Ariki-Koike algebra and let $\boldsymbol{\lambda} \in \Lambda^r$. The abacus configuration of $\boldsymbol{\lambda}$ with respect to $\boldsymbol{a}$ is the r-tuple of abacus configurations with kth component given by $(\lambda^{(k)}, a_k)$.

Abacus configurations for $\mathcal{H}_{r,n}(\boldsymbol{a})$

Definition

Let $\mathcal{H}_{r,n}(\boldsymbol{a})$ be an Ariki-Koike algebra and let $\boldsymbol{\lambda} \in \Lambda^r$. The abacus configuration of $\boldsymbol{\lambda}$ with respect to $\boldsymbol{a}$ is the r-tuple of abacus configurations with kth component given by $(\lambda^{(k)}, a_k)$.

Definition

Extend the map Φ_i to multipartitions by applying it to each component. Φ_i still maps blocks to blocks.

Say that $\Phi_i : B \to B'$ is a Scopes map if **for every** $\boldsymbol{\lambda} \in B$ the map Φ_i restricted to each component is a Scopes map.

Abacus configurations for $\mathcal{H}_{r,n}(\boldsymbol{a})$

Definition

Let $\mathcal{H}_{r,n}(\boldsymbol{a})$ be an Ariki-Koike algebra and let $\boldsymbol{\lambda} \in \Lambda^r$. The abacus configuration of $\boldsymbol{\lambda}$ with respect to $\boldsymbol{a}$ is the r-tuple of abacus configurations with kth component given by $(\lambda^{(k)}, a_k)$.

Definition

Extend the map Φ_i to multipartitions by applying it to each component. Φ_i still maps blocks to blocks.

Say that $\Phi_i : B \to B'$ is a Scopes map if **for every** $\boldsymbol{\lambda} \in B$ the map Φ_i restricted to each component is a Scopes map.

Define an equivalence relation $\sim_{\mathrm{Sc}}$ on the set of blocks of the symmetric group algebras by the closure of the relation $B \sim_{\mathrm{Sc}} B'$ if $B = \Phi_i(B')$ for Φ_i a Scopes map.

Scopes equivalence

Definition

Suppose that B is a $\sim$-equivalence class. Set

$$m = \max\{\text{hook}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in B\}, \qquad B_0 = \{\boldsymbol{\lambda} \in B \mid \text{w}(\boldsymbol{\lambda}) = m\}.$$

Scopes equivalence

Definition

Suppose that B is a $\sim$-equivalence class. Set

$$m = \max\{\mathrm{hook}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in B\}, \qquad B_0 = \{\boldsymbol{\lambda} \in B \mid \mathrm{w}(\boldsymbol{\lambda}) = m\}.$$

Proposition (Dell'Arciprete)

$\Phi_i : B \to B'$ is a Scopes map if for every $\boldsymbol{\lambda} \in B_0$ the map Φ_i restricted to each component is a Scopes map.

Scopes equivalence

Definition

Suppose that B is a $\sim$-equivalence class. Set

$$m = \max\{\mathrm{hook}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in B\}, \qquad B_0 = \{\boldsymbol{\lambda} \in B \mid \mathrm{w}(\boldsymbol{\lambda}) = m\}.$$

Proposition (Dell'Arciprete)

$\Phi_i : B \to B'$ is a Scopes map if for every $\boldsymbol{\lambda} \in B_0$ the map Φ_i restricted to each component is a Scopes map.

Theorem

Suppose the blocks B and B' are Scopes equivalent.

- *B and B' are decomposition equivalent (Dell'Arciprete).*
- *B and B' are Morita equivalent (Webster).*

Rouquier / RoCK blocks for Ariki-Koike algebras

Definition

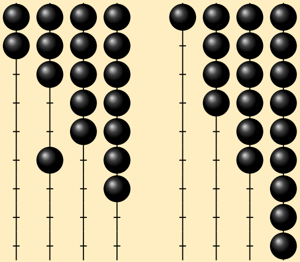
$\mathcal{H}_{r,n}(\boldsymbol{a})$ an Ariki-Koike algebra.

- $\boldsymbol{\lambda}$ is a Rouquier multipartition if $(\lambda^{(k)}, a_k)$ is a Rouquier partition for all $0 \leq k \leq r-1$.
- A $\sim$-equivalance class $\mathcal{R}$ is a Rouquier block if every $\boldsymbol{\lambda} \in \mathcal{R}$ is a Rouquier multipartition.
- A $\sim$-equivalance class $\mathcal{R}$ is a RoCK block if it is Scopes equivalent to a Rouquier block.

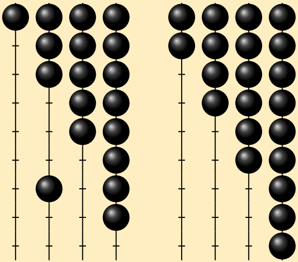
Rouquier multipartitions

Example

Rouquier



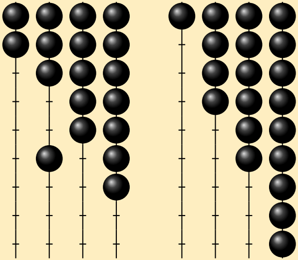
Not Rouquier



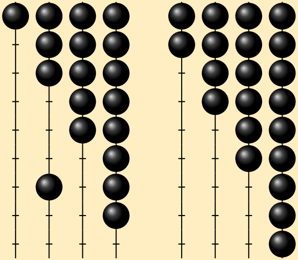
Rouquier multipartitions

Example

Rouquier



Not Rouquier



Lemma (Dell'Arciprete)

B is a Rouquier block if and only if every $\boldsymbol{\lambda} \in B_0$ is a Rouquier multipartition.

Work of Webster

Lemma (L.)

Say that a stretched Rouquier block is one in which $\mathfrak{b}^k_{i+1} - \mathfrak{b}^k_i \gg 0$ for all $0 \leq i < e-1$ and $0 \leq k \leq r-1$. Then any Rouquier block is Scopes equivalent to a stretched Rouquier block.

Setup (Webster)

$\mathcal{C}$ a categorical module over an affine Lie algebra $\mathfrak{g} \rightsquigarrow$ Scopes chambers $\rightsquigarrow$ RoCK chambers

Theorem (Webster)

For any categorical representation $\mathcal{C}$ of $\mathfrak{g} = \mathfrak{sl}_e$ with support $V(\Lambda)$, the Scopes equivalence classes will coincide those for the Ariki-Koike algebra, and a Scopes equivalence class is RoCK if and only if it contains a Rouquier weight.

Decomposition Numbers for Hecke algebras

Theorem (Leclerc-Miyachi, Chuang-Tan, James-L.-Mathas)

Let $r = 1$. Suppose that $\mathcal{R}$ is a Rouquier block, that $p = 0$ or weight $w < p$, and that $\lambda, \mu \in \mathcal{R}$ with μ e-regular. Then

$$[S^\lambda : D^\mu]_v = v^{\omega(\lambda)-\omega(\mu)} \sum_{\alpha_0,\dots,\alpha_e} \sum_{\beta_0,\dots,\beta_{e-1}} \prod_{i=0}^{e-1} c^{\mu_i}_{\alpha_i\beta_i} c^{\lambda_i}_{\beta_i(\alpha_{i+1})'}$$

where

$$|\alpha_i| = \sum_{j=0}^{i-1} \Big(|\lambda_j| - |\mu_j|\Big), \qquad |\beta_i| = |\lambda_i| + \sum_{j=0}^{i} \Big(|\mu_j| - |\lambda_j|\Big),$$

$$\omega(\lambda) - \omega(\mu) = \sum_{i=0}^{e-1} i\Big(|\mu_i| - |\lambda_i|\Big).$$

Decomposition numbers for Ariki-Koike algebras

Theorem [L.]

Suppose that $\boldsymbol{\lambda} \approx \boldsymbol{\mu}$ lie in a Rouquier block.

$$\boldsymbol{\lambda} \leftrightarrow ((\lambda^0_0, \lambda^0_1, \ldots, \lambda^0_{e-1}), \ldots, (\lambda^{r-1}_0, \lambda^{r-1}_1, \ldots, \lambda^{r-1}_{e-1})),$$
$$\boldsymbol{\mu} \leftrightarrow ((\mu^0_0, \mu^0_1, \ldots, \mu^0_{e-1}), \ldots, (\mu^{r-1}_0, \mu^{r-1}_1, \ldots, \mu^{r-1}_{e-1})).$$

Decomposition numbers for Ariki-Koike algebras

Theorem [L.]

Suppose that $\boldsymbol{\lambda} \approx \boldsymbol{\mu}$ lie in a Rouquier block.

$$\boldsymbol{\lambda} \leftrightarrow ((\lambda_0^0, \lambda_1^0, \ldots, \lambda_{e-1}^0), \ldots, (\lambda_0^{r-1}, \lambda_1^{r-1}, \ldots, \lambda_{e-1}^{r-1})),$$
$$\boldsymbol{\mu} \leftrightarrow ((\mu_0^0, \mu_1^0, \ldots, \mu_{e-1}^0), \ldots, (\mu_0^{r-1}, \mu_1^{r-1}, \ldots, \mu_{e-1}^{r-1})).$$

If $\boldsymbol{\mu}$ indexes a simple module $D^{\boldsymbol{\mu}}$ and $p = 0$ or $\mathrm{w}(\mu^{(k)}) < p$ for all k then

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_v = g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) := v^{\omega(\boldsymbol{\lambda}) - \omega(\boldsymbol{\mu})}$$
$$\sum_{\boldsymbol{\alpha} \in \Gamma_{e+1}^r} \sum_{\boldsymbol{\beta} \in \Gamma_e^r} \sum_{\boldsymbol{\gamma} \in \Gamma_e^{r+1}} \sum_{\boldsymbol{\delta} \in \Gamma_e^r} \left(\prod_{k=0}^{r-1} \prod_{i=0}^{e-1} c_{\mu_i^k \gamma_i^k}^{\delta_i^k} c_{\gamma_i^{k+1} \alpha_i^k \beta_i^k}^{\delta_i^k} c_{\beta_i^k (\alpha_{i+1}^k)'}^{\lambda_i^k} \right)$$

where $\gamma_0^0 = \ldots = \gamma_{e-1}^0 = \gamma_0^r = \ldots = \gamma_{e-1}^r = \emptyset$.

Example of decomposition numbers

Example

Let $r = 2$ and $e = 3$ so that

$$\boldsymbol{\lambda} \leftrightarrow ((\lambda_0^0, \lambda_1^0, \lambda_2^0), (\lambda_0^1, \lambda_1^1, \lambda_2^1)), \quad \boldsymbol{\mu} \leftrightarrow ((\emptyset, \mu_1^0, \mu_2^0)), (\emptyset, \mu_1^1, \mu_2^1)).$$

Then $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$ is equal to

$$\sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta},\\ \boldsymbol{\gamma},\boldsymbol{\delta}}} \begin{pmatrix} c^{\delta_0^0}_{\mu_0^0\gamma_0^0} & & c^{\delta_1^0}_{\mu_1^0\gamma_1^0} & & c^{\delta_2^0}_{\mu_2^0\gamma_2^0} & \\ c^{\delta_0^0}_{\alpha_0^0\beta_0^0\gamma_0^1} & c^{\lambda_0^0}_{\beta_0^0(\alpha_1^0)'} & c^{\delta_1^0}_{\alpha_1^0\beta_1^0\gamma_1^1} & c^{\lambda_1^0}_{\beta_1^0(\alpha_2^0)'} & c^{\delta_2^0}_{\alpha_2^0\beta_2^0\gamma_2^1} & c^{\lambda_2^0}_{\beta_2^0(\alpha_3^0)'} \\ c^{\delta_0^1}_{\mu_0^1\gamma_0^1} & & c^{\delta_1^1}_{\mu_1^1\gamma_1^1} & & c^{\delta_2^1}_{\mu_2^1\gamma_2^1} & \\ c^{\delta_0^1}_{\alpha_0^1\beta_0^1\gamma_0^2} & c^{\lambda_0^1}_{\beta_0^1(\alpha_1^1)'} & c^{\delta_1^1}_{\alpha_1^1\beta_1^1\gamma_1^2} & c^{\lambda_1^1}_{\beta_1^1(\alpha_2^1)'} & c^{\delta_2^1}_{\alpha_2^1\beta_2^1\gamma_2^2} & c^{\lambda_2^1}_{\beta_2^1(\alpha_3^1)'} \end{pmatrix}$$

where $\gamma_0^0 = \gamma_1^0 = \gamma_2^0 = \emptyset$ and $\gamma_0^2 = \gamma_1^2 = \gamma_2^2 = \emptyset$.

Ariki's Theorem

The Fock space

$\mathcal{F}^{\boldsymbol{a}}$ is the Fock space representation of $\mathcal{U} = \mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$.

- Basis $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda^r\}$.
- Canonical basis elements $G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \sim \boldsymbol{\mu}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}}$.

Ariki's Theorem

The Fock space

$\mathcal{F}^{\boldsymbol{a}}$ is the Fock space representation of $\mathcal{U} = \mathcal{U}_q(\hat{\mathfrak{sl}}_e)$.

- Basis $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda^r\}$.
- Canonical basis elements $G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \sim \boldsymbol{\mu}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}}$.

Ariki's Theorem

Suppose $p = 0$. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^r_n$ with $\boldsymbol{\mu}$ indexing a simple $\mathcal{H}$-module. Then

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_v = d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v).$$

Ariki's Theorem

The Fock space

$\mathcal{F}^{\boldsymbol{a}}$ is the Fock space representation of $\mathcal{U} = \mathcal{U}_q(\hat{\mathfrak{sl}}_e)$.

- Basis $\{s_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda^r\}$.
- Canonical basis elements $G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \sim \boldsymbol{\mu}} d_{\boldsymbol{\lambda\mu}}(v) s_{\boldsymbol{\lambda}}$.

Ariki's Theorem

Suppose $p = 0$. Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^r_n$ with $\boldsymbol{\mu}$ indexing a simple $\mathcal{H}$-module. Then

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_v = d_{\boldsymbol{\lambda\mu}}(v).$$

Theorem (L.)

Suppose that $\boldsymbol{\lambda} \approx \boldsymbol{\mu}$ lie in a Rouquier block with $\boldsymbol{\mu}$ e-regular. Then

$$d_{\boldsymbol{\lambda\mu}}(v) = g_{\boldsymbol{\lambda\mu}}(v).$$

We want to show:

Theorem

Suppose that $\boldsymbol{\mu}$ lies in a Rouquier block. Then

$$G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \approx \boldsymbol{\mu}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda} \sim \boldsymbol{\mu} \\ \boldsymbol{\lambda} \not\approx \boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}}.$$

Sketch of proof

$r = 1$

Case $r = 1$: Known by work of Leclerc and Miyachi.

Sketch of proof

$r = 1$

Case $r = 1$: Known by work of Leclerc and Miyachi.

Theorem (Fayers)

Suppose

$$\hat{\boldsymbol{\lambda}} = (\mu^{(1)}, \ldots, \lambda^{(r-1)}), \qquad \hat{\boldsymbol{\mu}} = (\mu^{(1)}, \ldots, \mu^{(r-1)}),$$
$$\boldsymbol{\lambda} = (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(r-1)}), \qquad \boldsymbol{\mu} = (\emptyset, \mu^{(1)}, \ldots, \mu^{(r-1)}).$$

Then

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) = d_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v).$$

Induction on r

Suppose $r > 1$

Suppose that $\boldsymbol{\mu} = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(r-1)}) \in \mathcal{R}$ and that $\mathrm{w}(\mu^{(0)}) = 0$. Let

$$\hat{\boldsymbol{\mu}} = (\mu^{(1)}, \ldots, \mu^{(r-1)}), \qquad \hat{\boldsymbol{\mu}}^{\emptyset} = (\emptyset, \mu^{(1)}, \ldots, \mu^{(r-1)}).$$

By the inductive hypothesis,

$$G(\hat{\boldsymbol{\mu}}) = \sum_{\boldsymbol{\lambda} \approx \hat{\boldsymbol{\mu}}} g_{\boldsymbol{\lambda}\hat{\boldsymbol{\mu}}}(v) s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda} \sim \hat{\boldsymbol{\mu}} \\ \boldsymbol{\lambda} \not\approx \hat{\boldsymbol{\mu}}}} d_{\boldsymbol{\lambda}\hat{\boldsymbol{\mu}}}(v) s_{\boldsymbol{\lambda}}.$$

Applying Fayers' result,

$$G(\hat{\boldsymbol{\mu}}^{\emptyset}) = \sum_{\substack{\boldsymbol{\lambda} \approx \hat{\mu}^{\emptyset} \\ \lambda^{(0)} = \emptyset}} g_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v) s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda} \sim \hat{\boldsymbol{\mu}}^{\emptyset}, \boldsymbol{\lambda} \not\approx \hat{\boldsymbol{\mu}}^{\emptyset} \\ \lambda^{(0)} = \emptyset}} d_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v) s_{\boldsymbol{\lambda}}.$$

Assume $w(\mu^{(0)}) = 0$

LLT induction

$$G(\hat{\boldsymbol{\mu}}^{\emptyset}) = \sum_{\substack{\boldsymbol{\lambda}\approx\hat{\mu}^{\emptyset} \\ \lambda^{(0)}=\emptyset}} g_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda}\sim\hat{\boldsymbol{\mu}}^{\emptyset},\boldsymbol{\lambda}\not\approx\hat{\boldsymbol{\mu}}^{\emptyset} \\ \lambda^{(0)}=\emptyset}} d_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v)s_{\boldsymbol{\lambda}}.$$

Use LLT induction to go from $\emptyset$ to $\mu^{(0)}$.

$$f(G(\hat{\boldsymbol{\mu}}^{\emptyset})) = \sum_{\substack{\boldsymbol{\lambda}\approx\boldsymbol{\mu} \\ \lambda^{(0)}=\mu^{(0)}}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda}\sim\boldsymbol{\mu},\boldsymbol{\lambda}\not\approx\boldsymbol{\mu} \\ \lambda^{(0)}=\mu^{(0)}}} d_{\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\mu}}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\tau}\sim\boldsymbol{\mu} \\ |\tau^{(0)}|<|\mu^{(0)}|}} b_{\boldsymbol{\tau}}(v)s_{\boldsymbol{\tau}}$$

$$\implies G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda}\approx\boldsymbol{\mu}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda}\sim\boldsymbol{\mu} \\ \boldsymbol{\lambda}\not\approx\boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v)s_{\boldsymbol{\lambda}}$$

Induction on w($\mu^{(0)}$)

Definition

Define

$$Q(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \approx \boldsymbol{\mu}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}}.$$

For $s > 0$ and $1 \leq j \leq e-1$, define

$$f^{(s,j)} = f_j^{(s)} \dots f_2^{(s)} f_1^{(s)} f_{j+1}^{(s)} \dots f_{e-1}^{(s)} f_0^{(s)} \in \mathcal{U}.$$

Induction on $\mathrm{w}(\mu^{(0)})$

Definition

Define

$$Q(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \approx \boldsymbol{\mu}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) s_{\boldsymbol{\lambda}}.$$

For $s > 0$ and $1 \leq j \leq e-1$, define

$$f^{(s,j)} = f_j^{(s)} \dots f_2^{(s)} f_1^{(s)} f_{j+1}^{(s)} \dots f_{e-1}^{(s)} f_0^{(s)} \in \mathcal{U}.$$

Lemma

If $\boldsymbol{\nu} \in \bar{R}^s$ then $f^{(s,j)} s_{\boldsymbol{\nu}}$ is a sum of terms $s_{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}$ is formed from $\boldsymbol{\nu}$ by moving beads down on runners $j-1$ and j on components of the abacus configuration of $\boldsymbol{\nu}$.

Action of $f^{(s,j)}$

Proposition

$$Q(\boldsymbol{\nu}) = \sum_{\boldsymbol{\lambda} \approx \boldsymbol{\nu}} g_{\boldsymbol{\lambda}\boldsymbol{\nu}}(v) s_{\boldsymbol{\lambda}}.$$

Then

$$f^{(s,j)} Q(\boldsymbol{\nu}) = \sum_{\Delta} c^{\Delta}_{\nu^0_j (1^s)} Q(\boldsymbol{\epsilon})$$

where $\boldsymbol{\epsilon}$ has quotient

$$(\nu^0_0, \ldots, \nu^0_{j-1}, \Delta, \nu^0_{j+1}, \ldots, \nu^0_{e-1}), (\nu^1_0, \nu^1_1, \ldots, \nu^1_{e-1}), \\ \ldots, (\nu^{r-1}_0, \nu^{r-1}_1, \ldots, \nu^{r-1}_{e-1})).$$

Final step

Proof

Take $\boldsymbol{\mu}$ with $\mathrm{w}(\boldsymbol{\mu}^{(0)}) > 0$. Form $\boldsymbol{\nu}$ by moving s beads up on runner j of the first component of $\boldsymbol{\mu}$. By the inductive hypothesis

$$G(\boldsymbol{\nu}) = \sum_{\boldsymbol{\lambda}\approx\boldsymbol{\nu}} g_{\boldsymbol{\lambda}\boldsymbol{\nu}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda}\sim\boldsymbol{\nu}\\ \boldsymbol{\lambda}\not\approx\boldsymbol{\nu}}} d_{\boldsymbol{\lambda}\boldsymbol{\nu}}(v)s_{\boldsymbol{\lambda}} = Q(\boldsymbol{\nu}) + \sum_{\substack{\boldsymbol{\lambda}\sim\boldsymbol{\nu}\\ \boldsymbol{\lambda}\not\approx\boldsymbol{\nu}}} d_{\boldsymbol{\lambda}\boldsymbol{\nu}}(v)s_{\boldsymbol{\lambda}}$$

$$\Longrightarrow \quad f^{(s,j)}G(\boldsymbol{\nu}) = \sum_{\Delta} c^{\Delta}_{\nu^0_j(1^s)} Q(\boldsymbol{\epsilon}) + \sum_{\boldsymbol{\lambda}\not\approx\boldsymbol{\mu}} r_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}$$

By induction, assume that $Q(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{\lambda}} g_{\boldsymbol{\lambda}\boldsymbol{\epsilon}}(v)s_{\boldsymbol{\lambda}}$ for $\boldsymbol{\epsilon} \neq \mu$. Then

$$G(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda}\approx\boldsymbol{\mu}} g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v)s_{\boldsymbol{\lambda}} + \sum_{\substack{\boldsymbol{\lambda}\sim\boldsymbol{\mu}\\ \boldsymbol{\lambda}\not\approx\boldsymbol{\mu}}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v)s_{\boldsymbol{\lambda}}.$$

Characteristic $p > 0$

Theorem

Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{R}$ with $\boldsymbol{\mu} \approx \boldsymbol{\lambda}$ where $\boldsymbol{\mu}$ indexes a Kleshchev multipartition and $\mathrm{w}(\mu^{(k)}) < p$ for all k. Then

$$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_v = g_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v).$$

Sketch of proof

- The theory of adjustment matrices gives a lower bound for the graded decomposition numbers $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]$.
- Try to repeat the proof for $p = 0$. Problems come when you look at $r \geq 1$ and $\mathrm{w}(\mu^{(0)}) > 0$. The multipartition $\boldsymbol{\nu}$ we defined before may not index a simple module.
- Work with the cyclotomic q-Schur algebra instead.

Gordon James (1945 – 2020)