Quantum Wreath Products and Schur-Weyl Duality

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GGOR Problem

Goal: Constructing highest weight covers for Hecke algebras.

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- Let W :complex reflection group.

$$\Rightarrow \begin{cases} \mathcal{H}_q(W) : \text{Hecke algebra} \overset{q\to 1}{=} \mathbb{C}W \\ \mathbb{H}_c(W) : \text{rational Cherednik algebra} \underset{\text{v.s.}}{=} S(\mathfrak{h}) \otimes \mathbb{C}W \otimes S(\mathfrak{h}^*) \end{cases}$$

Theorem ([Ginzburg-Guay-Opdam-Rouquier '03])

The Knizhnik-Zamolodchikov functor $\mathrm{KZ} : \mathcal{O}(\mathbb{H}_c(W)) \to \mathcal{H}_q(W)$-mod is a highest weight cover.

GGOR Conjecture ([Rouquier '08])

Under certain conditions on parameters, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(\mathbb{H}_c(\Sigma_d)) & \xrightarrow{\ \simeq \text{ as HWC}\ } & S_q(n, d)\text{-mod} \\ & {\scriptstyle \mathrm{KZ}} \searrow \quad \swarrow {\scriptstyle \mathrm{Schur}} & \\ & \mathcal{H}_q(\Sigma_d)\text{-mod} & \end{array}$$

GGOR Problem beyond type A

- Goal: For W beyond type A, make the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{O}(\mathbb{H}_c(W)) & \xrightarrow{\simeq} & S_q^W(n,d)\text{-mod} \\
 & \searrow^{\mathrm{KZ}} \qquad {}^{\mathrm{Schur}}\swarrow & \\
 & \mathcal{H}_q(W)\text{-mod} &
\end{array}
$$

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- Type B [Rouquier'08, Lai-Nakano-Xiang'22]
  Key step: construct Schur functor "via"

$$\mathcal{H}_q(B_d) \underset{\text{Morita}}{\approx} \prod_i \mathcal{H}_q(\Sigma_i \times \Sigma_{d-i}),$$

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$$\mathcal{H}_q(B_d) \underset{\text{Morita}}{\approx} \prod_i \mathcal{H}_q(\Sigma_i \times \Sigma_{d-i}),$$

- Type $G(p,p,d)$: Morita theorem is available [Hu'02, Hu-Mathas'12]. In particular, for type $D_d = G(2,2,d)$,

$$\mathcal{H}_q(D_d) \underset{\text{Morita}}{\approx} \begin{cases} \prod_i \mathcal{H}_q(\Sigma_i \times \Sigma_{d-i}) & \text{if } d = 2m+1; \\ \mathcal{H}_q(m \wr 2) \oplus \prod_i \mathcal{H}_q(\Sigma_i \times \Sigma_{d-i}) & \text{if } d = 2m, \end{cases}$$

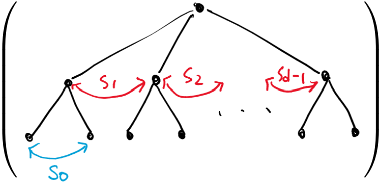
where $\mathcal{H}_q(m \wr 2)$ is a deformation of $\mathbb{C}\Sigma_m \wr \Sigma_2$ (to be discussed).

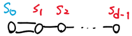
Wreath Products (For Groups)

- Recall that for any group G,

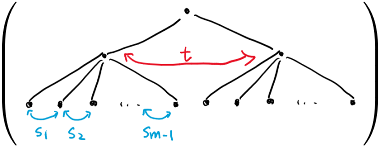
$$G \wr \Sigma_d := G^d \rtimes \Sigma_d \ni (g_1, \ldots, g_d)w \equiv w(g_{w(1)}, \ldots, g_{w(d)})$$

e.g.

$C_2 \wr \Sigma_d = \mathrm{Aut}$ 

$= W($ 

$)$

$\Sigma_m \wr \Sigma_2 = \mathrm{Aut}$ 

$\subseteq \Sigma_{2m}$

The Hu Algebra $\mathcal{H}_q(m \wr 2)$

- Let $\mathcal{H}_q(m \wr 2) := \langle \mathcal{H}_q(\Sigma_m \times \Sigma_m), h_m \rangle \subseteq \mathcal{H}_q(\Sigma_{2m})$ for some element h_m (to be defined) s.t. $\dim \mathcal{H}_q(m \wr 2) = \#(\Sigma_m \wr \Sigma_2)$.

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Hu's (type B) construction

- Let $\underset{T_0^2=1}{\mathcal{H}^{\mathrm{B}} := \mathcal{H}_{1,q}(B_d)} \supseteq \underset{T_i^2=(q-q^{-1})T_i+1}{\mathcal{H} := \mathcal{H}_q(\Sigma_d)}$.

  Define Jucys-Murphy elements $u_1^{\pm} := 1 \pm T_0$,
  $u_{i+1}^{\pm} := u_i^{\pm}(1 \pm T_i \dots T_1 T_0 T_1 \dots T_i)$.

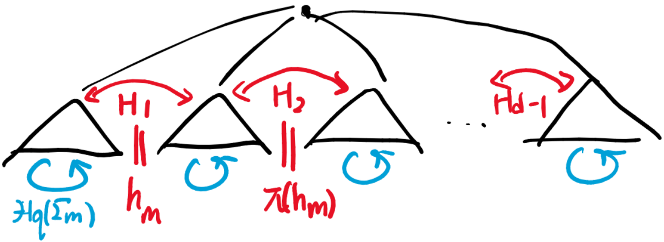
Proposition/Definition

1. There's a unique $h_m \in \mathcal{H}$ s.t. $u_m^- T_t u_m^+ \in h_m u_m^+ + \sum_{j>m} \mathcal{H} u_j^+ \mathcal{H}$.
2. There's a unique $z_m \in Z(\mathcal{H} \otimes \mathcal{H})$ s.t. $(u_m^- T_t u_m^+)^2 = z_m (u_m^- T_t u_m^+)$.
3. $h_m^2 = z_m$.

The generalized Hu Algebra $\mathcal{H}_q(m \wr d)$

- One can then define the generalized Hu algebra $\mathcal{H}_q(m \wr d)$ as below:
  Consider an assignment $\pi : T_i \mapsto T_{i+m}$.
  Let $H_1 := h_m$, $H_{i+1} := \pi(H_i)$, and let

$$\mathcal{H}_q(m \wr d) := \langle \mathcal{H}_q(\Sigma_m^d), H_1, \ldots, H_{d-1} \rangle \subseteq \mathcal{H}_q(\Sigma_{dm}).$$



⚠ It's almost impossible to use this constrction to investigate $\mathcal{H}_q(m \wr d)$. E.g., what's the relation between $h_m \otimes 1$ and $1 \otimes h_m$ in $\mathcal{H}_q(\Sigma_{3m})$?

The Hu Algebra $\mathcal{H}_q(m \wr 2)$

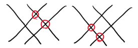
Our (type A) construction

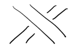
Theorem (Lai-Nakano-Xiang'23)

There is an explicit formula for h_m (and hence for z_m) without using Jucys-Murphy elements

Example ($m = 2$)

$t = (s_2 s_3)(s_1 s_2) \rightsquigarrow T_t = T_2 T_3 T_1 T_2$ has complicated powers



$h_m =$ $T_2 T_3 T_1 T_2$  $+$ $T_2 T_3 \overline{T_1}\,\overline{T_2}$  $+ \Big($ $\overline{T_2}\,\overline{T_3} T_1 T_2$  $+$ $\overline{T_2}\,\overline{T_3}\,\overline{T_1}\,\overline{T_2}$  $\Big) T_3^2$

$$
\begin{aligned}
z_m = \ & (q - q^{-1})^2 (q^4 + 2q^2 - 2 + 2q^{-2} + q^{-4}) T_1 T_3 \qquad \in Z(\mathcal{H} \otimes \mathcal{H}) \\
& + (q - q^{-1})(q^4 + 4q^2 - 2 + 4q^{-2} + q^{-4})(T_1 + T_3) \\
& + 2(q^4 + 4q^2 - 2 + 4q^{-2} + q^{-4})
\end{aligned}
$$

Presentation of the Hu Algebra

- The Hu algebra admits the following presentation:

$$\mathcal{H}_q(m \wr 2) = \frac{\langle \mathcal{H}_q(\Sigma_m)^{\otimes 2}, H_1 \rangle}{\begin{array}{l} H_1^2 = z_m \\ H_1(T_x \otimes T_y) = (T_y \otimes T_x)H_1 \end{array}}$$

$$h_m \mapsto H_1$$

$$T_i \mapsto \begin{cases} T_i \otimes 1 & \text{if } i < m, \\ 1 \otimes T_i & \text{if } i > m. \end{cases}$$

- We need a theory accomodating Heck-like algebras with quadratic relations for the form

$$H^2 = SH + R \quad \text{where } S, R \in B \otimes B \quad \text{for some base algebra } B.$$

Definition of Quantum Wreath Product

- Wreath product $G \wr S_n$
- Base group G
- Direct product G^n
- Braid relations $s_i s_j \ldots = s_j s_i \ldots$
- Quadratic relation
  $s^2 = 1$
- Wreath relation
  $s(g, h) = (h, g)s$

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- Quantum wreath product $B \wr \mathcal{H}(n)$
- Base Algebra B
- Tensor product $B^{\otimes n}$
- Braid relations $H_i H_j \ldots = H_j H_i \ldots$
- $H^2 = SH + R$ for $R, S \in B \otimes B$
- $H(a \otimes b) = \sigma(a \otimes b)H + \rho(a \otimes b)$ for $\rho, \sigma \in \mathrm{End}(B \otimes B)$

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- $H(a \otimes b) = \sigma(a \otimes b)H + \rho(a \otimes b)$
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Definition (Lai-Nakano-Xiang'23)

Given an algebra B, a choice of parameter $Q = (R, S, \rho, \sigma)$, and a rank n, we define a quantum wreath product that produces an algebra $B \wr \mathcal{H}(n)$ generated by $B^{\otimes n}$ and $H_1, \ldots, H_{n-1}$ subject to the the above local relations for all $1 \leq i \leq n-1$.

Popositions/Examples

Hu algebra $\mathcal{H}_q(m \wr 2)$ $\simeq \mathcal{H}_q(\Sigma_m) \wr \mathcal{H}(2)$	$H_1^2 = z_m$ $H_1(T_x \otimes T_y) = (T_y \otimes T_x)H_1$	$\Sigma_m \wr \Sigma_2$
Affine Hecke algebra $\mathcal{H}_q^{\mathrm{aff}}(\Sigma_n)$ $\simeq K[X^{\pm 1}] \wr \mathcal{H}(n)$	$H^2 = (q - q^{-1})(1 \otimes 1)H + 1 \otimes 1$ $H(X \otimes 1) = (1 \otimes X)H - (q - q^{-1})X \otimes 1$	$\mathbb{Z} \wr \Sigma_n = \Sigma_n^{\text{ext}}$
Ariki-Koike $\mathcal{H}_{q,\vec{v}}(C_m \wr \Sigma_n)$ $\simeq \frac{K[X^{\pm 1}] \wr \mathcal{H}(n)}{\prod_{i=1}^m (X - q_i)}$	Same parameters	$C_m \wr \Sigma_n = G(m, 1, n)$
degenerate AHA $\mathcal{H}^{\deg}(\Sigma_n)$ $\simeq K[X] \wr \mathcal{H}(n)$	$H^2 = 1 \otimes 1$ $H(X \otimes 1) = (1 \otimes X)H - 1 \otimes 1$	$\mathbb{N} \wr \Sigma_n$
Wan-Wang's $\mathcal{H}^{\deg}(G \wr \Sigma_n)$ $\simeq K[X]G \wr \mathcal{H}(n)$	$H^2 = 1 \otimes 1$ $H(X \otimes 1) = (1 \otimes X)H - \sum_g g \otimes g^{-1}$	$G \wr \Sigma_n$

More Examples

Our quantum wreath product is a nautral generalization of

- [Savage'20, Rosso-Savage'20] (affine) Frobenius Hecke algebras, and quantum wreath product algebra,
- [Kleshchev-Muth'19] rank n affinization algebra.

Just to name some more algebras that can be produced from taking quantum wreath products:

- Affine Hecke algebras of type C
- Evseev-Kleshchev's super wreath product algebra
- Kleshchev-Muth's Affine zigzag algebras
- (Affine) Yokonuma-Hecke algebras
- ...

Structure and Representation Theory

- We consider the following upgrade:

Basis of B	$\Longrightarrow$	basis of $A := B \wr \mathcal{H}(d)$
Symmetric algebra structure of B	$\Longrightarrow$	Symmetric algebra structure of A
Schur duality for B $S^B \curvearrowright T_B \curvearrowleft B$	$\Longrightarrow$	Schur duality for A $S^A \curvearrowright T_A \curvearrowleft A$
$\vdots$		$\vdots$
1-cover of B	$\Longrightarrow$	1-cover of A

Basis Theorem

Theorem 1 (Lai-Nakano-Xiang'23)

The neccesary and sufficient conditions such that the basis $\{b_i\}_{i\in I}$ can be upgraded to the basis $\{(b_{i_1} \otimes \cdots \otimes b_{i_d})H_w \mid i_j \in I, w \in \Sigma_d\}$ of A are:

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for $\{i,j\}=\{1,2\}$, $a,b\in B\otimes B$:

$$\sigma(1\otimes 1)=1\otimes 1,\quad \rho(1\otimes 1)=0, \tag{1}$$
$$\sigma(ab)=\sigma(a)\sigma(b),\quad \rho(ab)=\sigma(a)\rho(b)+\rho(a)b, \tag{2}$$
$$\sigma(S)S+\rho(S)+\sigma(R)=S^2+R,\quad \rho(R)+\sigma(S)R=SR, \tag{3}$$
$$r_S\sigma^2+\rho\sigma+\sigma\rho=r_S\sigma,\quad r_R\sigma^2+\rho^2=r_S\rho+r_R, \tag{4}$$
$$\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2,\quad \rho_i\sigma_j\sigma_i=\sigma_j\sigma_i\rho_j, \tag{5}$$
$$\rho_i\sigma_j\rho_i=r_{S_j}\sigma_j\rho_i\sigma_j+\rho_j\rho_i\sigma_j+\sigma_j\rho_i\rho_j, \tag{6}$$
$$\rho_1\rho_2\rho_1+r_{R_1}\sigma_1\rho_2\sigma_1=\rho_2\rho_1\rho_2+r_{R_2}\sigma_2\rho_1\sigma_2, \tag{7}$$
$$S_1=\sigma_2\sigma_1(S_2),\quad R_1=\sigma_2\sigma_1(R_2),\quad \rho_2\sigma_1(S_2)=0=\rho_2\sigma_1(R_2), \tag{8}$$
$$\sigma_2\rho_1(S_2)S_2+\rho_2\rho_1(S_2)+\sigma_2\rho_1(R_2)=0=\rho_2\rho_1(R_2)+\sigma_2\rho_1(S_2)R_2, \tag{9}$$

where $X_i:=X^{(i,i+1)}$ and r_X is right multiplication by X.

Basis Theorem

Theorem 2 (Lai-Nakano-Xiang'23)

The neccesary and sufficient conditions such that the basis $\{b_i\}_{i\in I}$ can be upgraded to the basis $\{(b_{i_1} \otimes \cdots \otimes b_{i_d})H_w \mid i_j \in I, w \in \Sigma_d\}$ of A are:

for $\{i,j\} = \{1,2\}$, $a, b \in B \otimes B$:

$$\sigma(1\otimes 1) = 1\otimes 1, \quad \rho(1\otimes 1) = 0, \tag{1}$$

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \rho(ab) = \sigma(a)\rho(b) + \rho(a)b, \tag{2}$$

$$\sigma(S)S + \rho(S) + \sigma(R) = S^2 + R, \quad \rho(R) + \sigma(S)R = SR, \tag{3}$$

$$r_S\sigma^2 + \rho\sigma + \sigma\rho = r_S\sigma, \quad r_R\sigma^2 + \rho^2 = r_S\rho + r_R, \tag{4}$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \rho_i\sigma_j\sigma_i = \sigma_j\sigma_i\rho_j, \tag{5}$$

$$\rho_i\sigma_j\rho_i = r_{S_j}\sigma_j\rho_i\sigma_j + \rho_j\rho_i\sigma_j + \sigma_j\rho_i\rho_j, \tag{6}$$

$$\rho_1\rho_2\rho_1 + r_{R_1}\sigma_1\rho_2\sigma_1 = \rho_2\rho_1\rho_2 + r_{R_2}\sigma_2\rho_1\sigma_2, \tag{7}$$

$$S_1 = \sigma_2\sigma_1(S_2), \quad R_1 = \sigma_2\sigma_1(R_2), \quad \rho_2\sigma_1(S_2) = 0 = \rho_2\sigma_1(R_2), \tag{8}$$

$$\sigma_2\rho_1(S_2)S_2 + \rho_2\rho_1(S_2) + \sigma_2\rho_1(R_2) = 0 = \rho_2\rho_1(R_2) + \sigma_2\rho_1(S_2)R_2, \tag{9}$$

where $X_i := X^{(i,i+1)}$ and r_X is right multiplication by X.

These conditions (1)–(9) can be simplified, when $H(a\otimes b) = (b\otimes a)H$, to

$$R = \sigma(R), \quad (\sigma(S) - S)R = 0, \quad r_S(\mathrm{id} - \sigma) = 0.$$

Symmetric Algebra Structures

Proposition (Lai-Nakano-Xiang'23)

Let B be a symmetric algebra with trace map $\mathrm{tr} : B \to K$. Then $A = B \wr \mathcal{H}(d)$ is also a symmetric algebra if

$$\mathrm{tr}(a \otimes b) = \mathrm{tr}(\sigma(a \otimes b)), \quad \rho = 0, \quad R \text{ is invertible.}$$

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Since symmetric algebras are self-injective, we seek to apply the following theorem to obtain Schur-Weyl duality for our quantum wreath product algebra $A = B \wr \mathcal{H}(d)$:

Theorem (Curtis-Reiner'62)

Assume that A is a finite dimensional self-injective algebra. If T is a faithful A-module, then $\mathrm{End}_S(T) = A$, where $S := \mathrm{End}_A(T)$.

A Splitting Lemma

Lemma

If $B^{\oplus d} \overset{\oplus}{\subseteq} T_B$, then $T_A := T_B^{\otimes d}$ is a faithful $B \wr \mathcal{H}(d)$-module.

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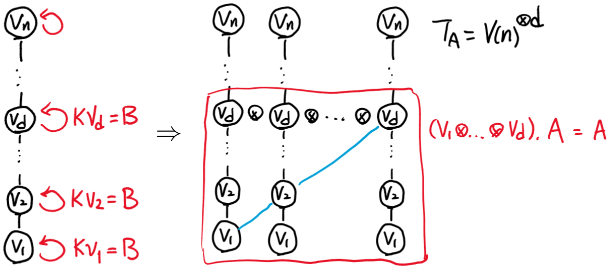
If $B^{\oplus d} \overset{\oplus}{\subseteq} T_B$, then $T_A := T_B^{\otimes d}$ is a faithful $B \wr \mathcal{H}(d)$-module.

Let $B = K$, $T_B = V(n) := \oplus_{i=1}^n K v_i$.

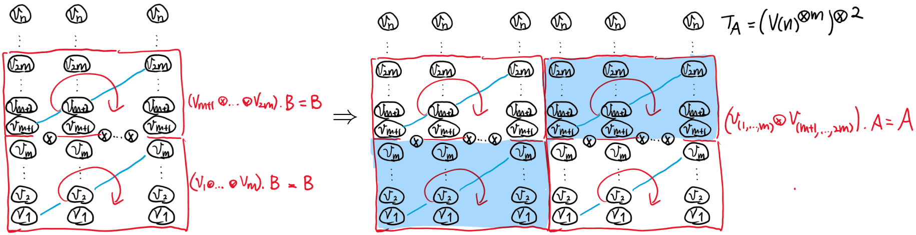


Corollary

If $n \geq d$, then Schur-Weyl duality holds for $\mathcal{H}_q(\Sigma_d) = K \wr \mathcal{H}(d)$.

A Splitting Lemma

Let $B = \mathcal{H}_q(\Sigma_m)$, $T_B = V(n)^{\otimes m}$.



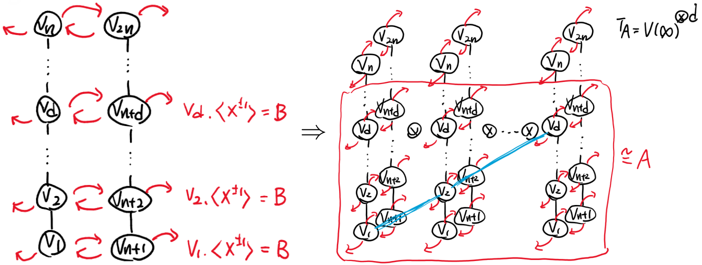
Corollary

If $n \geq 2m$, then Schur-Weyl duality holds for $\mathcal{H}_q(m \wr 2) = \mathcal{H}_q(\Sigma_m) \wr \mathcal{H}(2)$.

Our result, when applied to super wreath product algebras, recovers the Schur duality in [Evseev-Kleshchev'17] which was used in their proof of Turner's conjecture.

Related Schur-Weyl Dualities

Let $B = K[X^{\pm 1}]$, $T_B = V(\infty) := \oplus_{i\in\mathbb{Z}} K v_i$ on which $X^{\pm 1} : v_i \mapsto v_{i\pm n}$. Then the splitting lemma holds (while both A, B are not symmetric algebras).

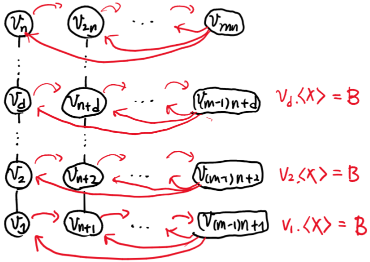


Theorem (Chari-Pressley'94, Green'99)

If $n \geq d$, then Schur-Weyl duality holds for $\mathcal{H}_q^{\mathrm{aff}}(\Sigma_d) = K[X^{\pm 1}] \wr \mathcal{H}(d)$.

Related Schur-Weyl Dualities

Let $B = K[X]/\prod_{i=1}^{m}(X - q_i)$, $T_B = V(mn)$ on which $X : v_j \mapsto v_{j+n}$ for $j < m(n-1)$. Then the splitting lemma holds (while the symmetric algebra structure is not obtained from our approach).



Corollary (James-Mathas'00)

If $n \geq d$, then Schur-Weyl duality holds for $\mathcal{H}_{q,\vec{q}}(C_m \wr \Sigma_d)$.

Similar remarks can be made regarding affine Hecke algebras of type C.

Canonical Basis For Hu Algebra....?

- Let $b_1 := h_m \overline{T}_{w_\circ(\Sigma'_m)}$. Then the Hu algebra admits a bar-invariant basis $\{b_w\}_w$ built from b_1 such that

$$b_w \in T_w + \sum_{x<w} q\mathbb{N}[q]\, T_x.$$

This bar-invariant basis is not canonical because

$$b_1^2 \notin \sum_w q\mathbb{N}[q]\, b_w.$$

- However, b_1 seems to admit a positive expansion in the dual canonical basis $\{c_w \mid w \in \Sigma_{2m}\}$, i.e.,

$$b_1 \in \sum_{w\in\Sigma_{2m}} \mathbb{N}[q^{\pm 1}]\, c_w,$$

where the dual basis element c_w is characterized by

$$\overline{c}_w = c_w \in T_w + \sum_{y<w} (-q^{-1})\mathbb{N}[-q^{-1}]\, T_y,$$

and it satisfies that $c_x c_y \in \sum_z \mathbb{N}[-q^{\pm 1}]\, c_z$.

Canonical Basis For Hu Algebra....?

- For $m = 1$,

$$b_1 = 2c_{s_1} + (q + q^{-1}).$$

- For $m = 2$,

$$\begin{aligned} b_1 &= 4c_{2.3.1.2.3} \\ &+ 2(q + q^{-1})(c_{2.3.1.2} + c_{2.1.2.3} + c_{3.1.2.3}) \\ &+ 4(c_{2.1.2} + c_{3.1.2}) + 2(q^2 + q^{-2})c_{1.2.3} \\ &+ 2(q + q^{-1})(c_{2.1} + c_{1.2} + c_{3.2} + c_{2.3}) \\ &+ (q^2 + 2 + q^{-2})(c_1 + 2c_2 + c_3) \\ &+ (q^3 + q + q^{-1} + q^{-3}). \end{aligned}$$

- There seems to be an explanation via categorification

Thank you