Let $k = \mathbb{Q}$

Given $\lambda \vdash d$, let $Std(\lambda)$ = standard tableaux of shape λ.

When $\lambda = (n,n)$, $d = 2n$, there are two interesting combinatorial basis for the Specht module S^λ:

- $\{ v_T \mid T \in Std(\lambda) \}$ when S_d acts on the entries of T

- The web basis of noncrossing pairings of $1, 2, \ldots, 2n$

  $\{ w \mid w \text{ as above} \}$ with action given by the Skein relation

<u>Russell - Tymoczko</u> gave a bijection

$$\varphi : Std(\lambda = (n,n)) \to \{ 2n\text{-webs} \}_n$$

and a partial order, $\leq$, on webs

s.t.

Thm (Russell – Tymoczko, Rhoades, Im – Zhu, Hwang – Jang – Oh)

For all $T \in \mathrm{Std}(\lambda = (n,n))$

$$v_T = \varphi(T) + \sum_{w < \varphi(T)} c_{Tw}\, w$$

where $c_{Tw} \in \mathbb{Z}_{\geq 0}$

Thm (Heard – K.) Let $k = \mathbb{Q}(\varepsilon)$,

Let $\lambda = (a,b)$, let

$\{ v_T \mid T \in \mathrm{Std}(\lambda) \}$ be the standard basis

$\{ w \mid w$ an $(a+b, a-b))$ – web $\}$ of non crossing pairing of $1, 2, \dots, a+b, 1', 2', \dots, (a-b)' \}$

Then there is a bijection

$$\varphi : \mathrm{Std}(\lambda = (a,b)) \to \{ \text{webs} \}$$

and partial order $\leq$ on webs

such that

$$v_T = v_{\varphi(T)} + \sum_{w < \varphi(T)} c_{Tw}\, w$$

$$c_{Tw} \in \mathbb{Z}_{\geq 0}[q]$$

<u>Remark!</u> It's known that the web basis essentially corresponds to the Kazhdan–Lusztig when λ has two rows.

- So unitriangularity follows from McDonough–Pallikaros
- Positivity presumably follows from the known categorifications of $\mathfrak{sl}_2$ by Khovanov, Khovanov–Stroppel–Mazorchuk

The point here is to give a direct combinatorial proof.

Let $k = \mathbb{Q}(q)$, $H_d(q)$ finite Hecke algebra with generators $T_1, \dots, T_{d-1}$

$$\{ T_w := T_{s_1} \cdots T_{s_k} \mid w = s_1 \cdots s_k \text{ a reduced expression} \}$$

Let $\sigma_\lambda \in S_d$ be given by

1	2	3	4	5	6	7	k
k+1	k+2			k+l			

$\cdot\ \sigma_\lambda =$

1	3	5	7	9	...
2	4	6	8	10	

Let $D_\lambda = \{ x \in S_d \mid x \leq \sigma_\lambda \}$ ($\leq$: weak (right) Bruhat order)

Then $x \longmapsto$

1	3	5	7	...
2	4	6	8	

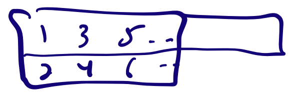
$\cdot\ x$

gives a bijection $D_\lambda \longrightarrow \mathrm{Std}(\lambda)$.

So the standard basis for S^λ:

$$\{ v_T = z_\lambda \cdot T_x \mid x \mapsto T \text{ under bijection} \}$$

$$z_\lambda = \Big[\sum_{\substack{w \\ \text{stabilizer}}} q^{\ell(w)} T_w\Big]\Big[\sum_{\substack{w \\ \text{column stabilizer of}}} (-q)^{-\ell(w)} T_w\Big]$$

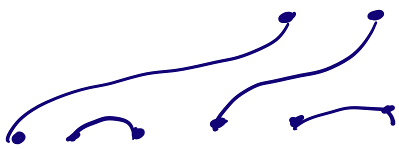


Web Basis !

Given $\lambda = (a,b)$, a $\underline{(a+b,\, a-b)}$-web is a non crossing pairing of $1, 2, \ldots, a+b$, $1', 2', \ldots, (a-b)'$

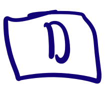
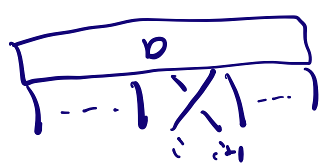
s.t. $1', \ldots, (a-b)'$ can't be paired with each other.

ⓔⓧ $a = 4 \quad b = 2$



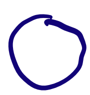
Then $S^{\lambda=(a,b)}$ has basis $\{ w \mid w \text{ } (a+b, a-b)\text{-web} \}$

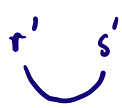
action is given by

 $. T_i =$ 

subject to the relations

① 

②  $= -[2]_q = -(q+q^{-1})$

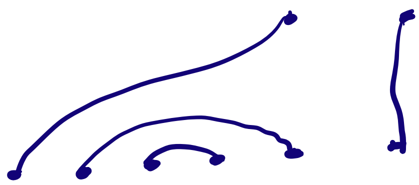
③ 

$=$ zero.

Bijection: $\varphi : Std(\lambda = (a,b)) \longrightarrow$ words

actually easier to show the inverse!

ex



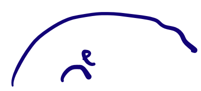
$\longmapsto$

1	2	3	6
4	5		

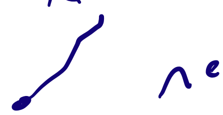
Partial order:

Def: nesting number. given an arc e in w

nest(e) = # arcs which "cover" e

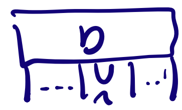


\+ # through strings to the left of e



Then $nest(w) = \sum_{e \text{ arc of } w} nest(e)$

nest() = 1 + 2 = 3

Key observation : $\text{nest}\Big($  $\Big) \leq \text{nest}(D) + 1$

This easily leads to

$$V_T = V_{\varphi(T)} + \sum_{\text{nest}(w) < \text{nest}(\varphi(T))} c_{Tw}\, w$$

But the real partial order is

$w \leq_i w' \iff w' =$ 

and $\text{nest}(w') = \text{nest}(w) + 1$

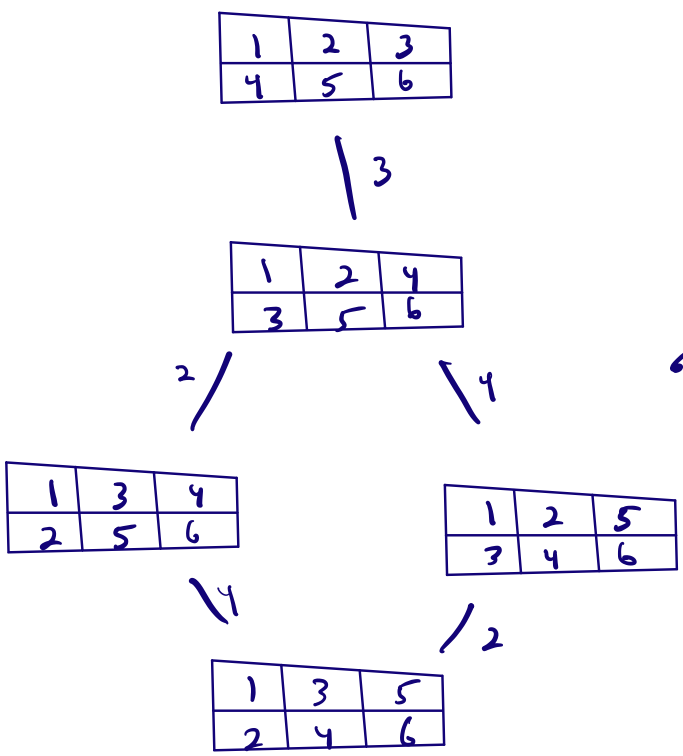
and $w \leq w'$ iff

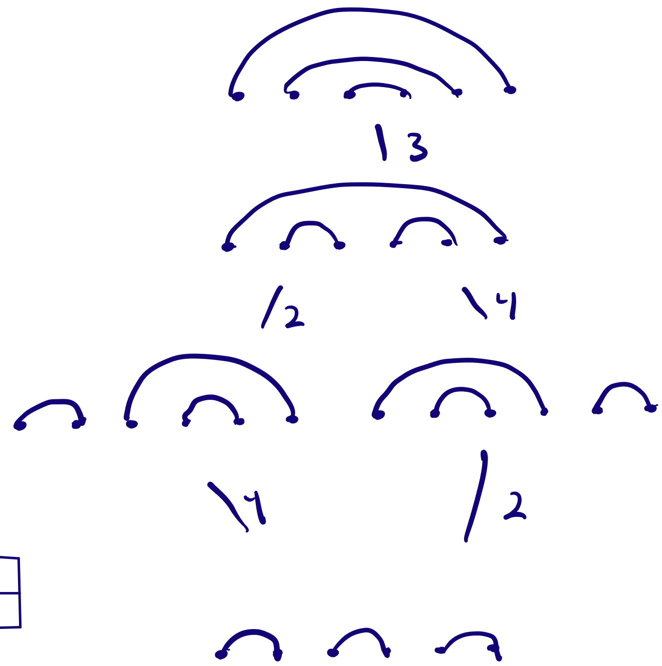
$$w \leq_{i_1} w_1 \leq \dots \leq_{i_k} w_{i_k} = w'$$

Then

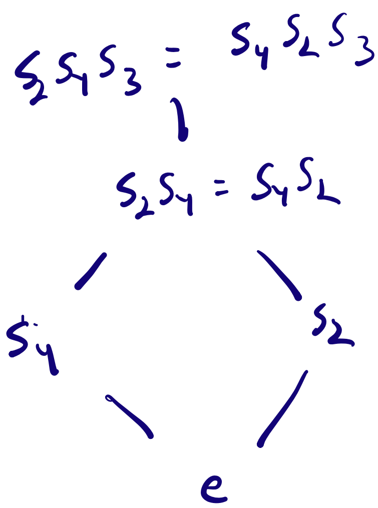
$$V_T = V_{\varphi(T)} + \sum_{w \leq \varphi(T)} c_{Tw}\, w$$

ex $\lambda = (3,3)$





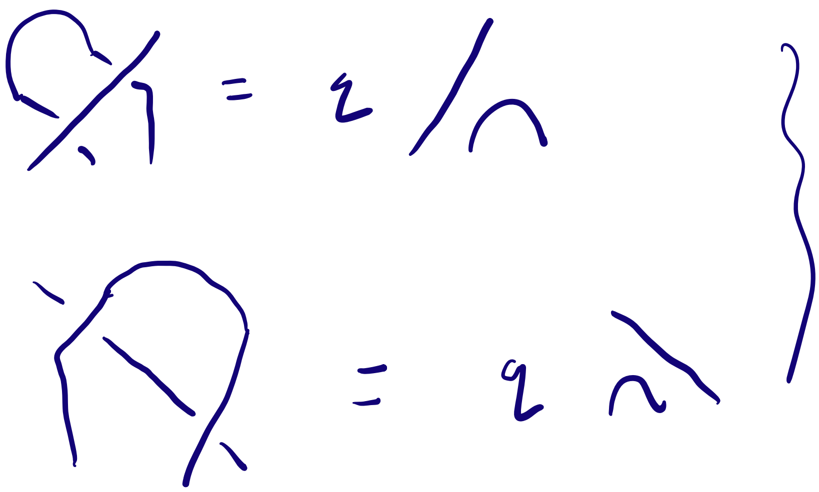
$\mathcal{Q}_\lambda$



Positivity:

Key observation: Avoid Bubbles?

Lemma: From the skein relation:



Reidemeister II

Proof:

- Show $z_\lambda = v_{\boxed{\begin{smallmatrix}1&3&5&\cdots\\2&4&6&\cdots\end{smallmatrix}}} = w_0 := \cap\,\cap \cdots \cap$

  and this is the unique minimal basis element on [illegible]

- So $v_T = z_\lambda \cdot T_{i_1} \cdots T_{i_k} = w_0 \cdot T_{i_1} \cdots T_{i_k}$

- Look up an algorithm to use the skein relation + Reidemeister II to avoid ever creating a bigon.

- Practice!