A geometric realization of Catalan functions

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Set up

We consider only $G = GL(n, \mathbb{C})$ and its rational representations.

Hence, irreducible rational representations are parametrized by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$$

such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

We refer it as V_λ, and it is a polynomial representation $\Leftrightarrow \lambda_n \geq 0$. (I.e. when λ is a partition)

If we write $B \subset G$ the subgroup of upper-trigngular matrices, then V_λ has a unique B-eigenvector $\mathbf{v}_\lambda$ on which

$$B \to B/[B,B] =: H \cong (\mathbb{C}^\times)^n$$

acts by

$$(\mathbb{C}^\times)^n \ni (X_1, \ldots, X_n) \mapsto X_1^{\lambda_1} \cdots X_n^{\lambda_n}.$$

Schur polynomials and Hall-Littlewood polynomials

The character of V_λ with respect to the action of $(\mathbb{C}^\times)^n$ is the Schur polynomial

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(\frac{X^\lambda}{\prod_{i<j}(1 - X_i^{-1}X_j)}\right) \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathfrak{S}_n},$$

where $X^\lambda := X_1^{\lambda_1} \cdots X_n^{\lambda_n}$ for $\lambda \in \mathbb{Z}^n$.

V_λ is polynomial representation $\Leftrightarrow s_\lambda \in \mathbb{Z}[X_1, \ldots, X_n]$.

Hall-Littlewood polynomial is its variant

$$HL_\lambda := c_\lambda \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(X^\lambda \frac{\prod_{i<j}(X_i - qX_j)}{\prod_{i<j}(X_i - X_j)}\right) \in \mathbb{Z}[q^{\pm 1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

with a normalization factor $c_\lambda \in \mathbb{Q}(q)$.

Orthogonality relations

We have an inner product on $\mathbb{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ such that

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

We have

$$\left\langle HL_\lambda, HL_\mu^\vee \right\rangle = \delta_{\lambda,\mu}$$

by regarding q as a scalar, where we have

$$HL_\lambda^\vee := \sum_{\sigma \in \mathfrak{S}_n} \sigma\left(\frac{X^\lambda \cdot X_1^{2(n-1)} \cdots X_{n-1}^2}{\prod_{i<j}((X_i - X_j)(X_i - qX_j))}\right) \in \mathbb{Z}[\![q, X_1^{\pm 1}, \dots, X_n^{\pm 1}]\!].$$

In order to consider things within $\mathbb{Z}[X_1, \dots, X_n]^{\mathfrak{S}_n}$, we consider the truncation operator

$$\left[\sum_\lambda a_\lambda s_\lambda\right] := \sum_{\lambda, \lambda_n \geq 0} a_\lambda s_\lambda \qquad a_\lambda \in \mathbb{C}(\!(q)\!).$$

This yields $\left\langle [HL_\lambda], \left[HL_\mu^\vee\right] \right\rangle = \delta_{\lambda,\mu}$.

Borel-Weil theorem

On $X := G/B$ (the flag variety) and $\lambda \in \mathbb{Z}^n$, we set

$$L_\lambda := \{(g, v) \in G \times \mathbb{C}\}/\sim$$

where $(g, v) \sim (gb, \chi_\lambda(b)v)$ for all $b \in B$ and $\chi_\lambda : B \to \mathbb{C}^\times$ is a character (slightly twisted from λ). This is a fiber bundle on X whose fiber is $\mathbb{C}$ (line bundle).

Theorem (Borel-Weil)

When $\lambda = (\lambda_i) \in \mathbb{Z}^n$ satisfies $\lambda_1 \geq \cdots \geq \lambda_n$, we have

$$V_\lambda = \Gamma(X, L_\lambda) = \{s : X \to L_\lambda \mid \pi \circ s = \mathrm{id}\},$$

where $\pi : L_\lambda \to X$ is the projection map.

(We neglect dualities here in order to save our memory.)

Geometric interpretation of $HL_\lambda^\vee$

On $X := G/B$ (the flag variety) and $\lambda \in \mathbb{Z}^n$, we set

$$L_\lambda := \{(g,v) \in G \times \mathbb{C}\}/\sim$$

where $(g,v) \sim (gb, \chi_\lambda(b)v)$ for all $b \in B$ and $\chi_\lambda : B \to \mathbb{C}^\times$ is a character (obtained from λ).

We pullback L_λ from X to T^*X (cotangent bundle) and denote by $\widetilde{L}_\lambda$. Then, we have

Theorem (R.Brylinski, Broer)

When $\lambda = (\lambda_i) \in \mathbb{Z}^n$ satisfies $\lambda_1 \geq \cdots \geq \lambda_n$, we have

$$HL_\lambda^\vee = \mathrm{char}_q \Gamma(T^*X, \widetilde{L}_\lambda) = \{s : T^*X \to \widetilde{L}_\lambda \mid \pi \circ s = \mathrm{id}\},$$

where $\pi : \widetilde{L}_\lambda \to T^*X$ is the projection map, and q counts the character coming from the $\mathbb{C}^\times$-action on the fibers of T^*X.

A naive question

We have

$$HL_0^\vee = \mathrm{char}_q\, \mathbb{C}[T^*X] = \Big(\prod_{m=1}^{n} \frac{1-q^m}{1-q}\Big) \cdot \Big(\prod_{1\le i,j\le n, i\neq j} \frac{1}{1-qX_iX_j^{-1}}\Big),$$

and hence

$$\left[HL_0^\vee\right] = 1 = \mathrm{char}_q\, \mathbb{C}[\overline{T^*X}],$$

where $\overline{T^*X}$ is *any* $G\times\mathbb{C}^\times$ equivariant compactification of $\mathcal{N}$ by Liouville's theorem (i.e. globally defined rational function on a compact connected variety is constant).

Problem

Can you provide nice $\overline{T^*X}$ *and line bundles* $\widetilde{L}_\lambda^+$ *on* $\overline{T^*X}$ *such that*

$$\left[HL_\lambda^\vee\right] = \mathrm{char}_q \Gamma(\overline{T^*X}, \widetilde{L}_\lambda^+) = \{s : \overline{T^*X} \to \widetilde{L}_\lambda \mid \pi\circ s = \mathrm{id}\}$$

and $\widetilde{L}_\lambda^+$ *restricts to* $\widetilde{L}_\lambda$ *on* T^*X.

Chen-Haiman's proposal

The cotangent bundle T^*X admits a vector subbundle $T^*_\Psi X$ (equipped with $G \times \mathbb{C}^\times$-action) for each Dyck path of size n:

Precisely, "upside-down" of a Dyck path defines a B-submodule of the fiber of T^*X at B/B, the strictly upper triangular matrices:

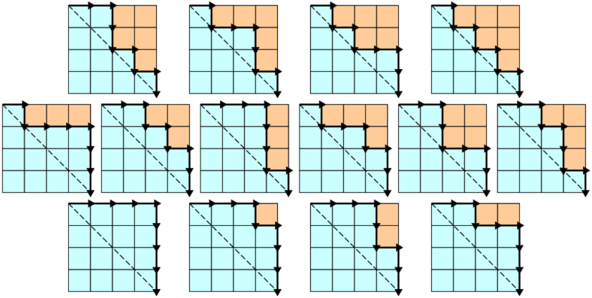

Figure: Dyck path of size 4 from Wikipedia (upside down) CC BY-SA 3.0
`https://en.wikipedia.org/wiki/Catalan_number`

Catalan polynomials and the Chen-Haiman conjecture

The line bundle $\widetilde{L}_\lambda$ on T^*X restricts to $T^*_\Psi X$ for each Dyck path.

Definition (Catalan polynomials)

For each Dyck path Ψ and a partition $\lambda \in \mathbb{Z}^n$, we set

$$HL_\lambda^\Psi := \left[\mathrm{char}_q \, \Gamma(T^*_\Psi X, \widetilde{L}_\lambda) \right] \in \mathbb{Z}[q, X_1, \ldots, X_n]^{\mathfrak{S}_n}.$$

We claim that this defines a nice family of symmetric functions. However, the RHS is guaranteed to be calculated by an explicit algebraic formula whenever the following holds:

Conjecture (Chen-Haiman's vanishing conjecture (2010))

For each Dyck path Ψ and a partition $\lambda \in \mathbb{Z}^n$, we have

$$H^{>0}(T^*_\Psi X, \widetilde{L}_\lambda) = \{0\}.$$

Little about Catalan polynomials

Even without the vanishing conjecture, we can algebraically define HL^{Ψ}_{λ} (through a formal application of the Weyl character formula) and study them. Some of the reasons we care about this is:

- It contains various generalizations of Kostka polynomials from the combinatorial study of some physics model from 1990s; (recall that the original Kostka polynomials form the transition matrix between $HL_{\bullet}/HL^{\vee}_{\bullet}$ and $s_{\bullet}$)
- It contains the k-Schur polynomials introduced by LaPonte-Lascoux-Morse, in connection with the structure of Macdonald polynomials (of type A);
- k-Schur polynomial is a basic ingredient of the study of the quantum cohomology of flag manifolds (of type A);
- Some of these features (that were conjectures), as well as the combinatorial part of the Chen-Haiman conjecture, are established by Blasiak-Morse-Pun-Summers.

The nilpotent cone

We can also enhance T^*X by enlarging the fiber (that is B-stable) into a G-module $\mathrm{Mat}(n, \mathbb{C})$ with adjoint action. Then, we have

$$T^*X \subset G \times_B \mathrm{Mat}(n, \mathbb{C}) \cong G/B \times \mathrm{Mat}(n, \mathbb{C}) \xrightarrow{\mathrm{pr}_2} \mathrm{Mat}(n, \mathbb{C}).$$

Its image is the space $\mathcal{N}$ of nilpotent matrices in $\mathrm{Mat}(n, \mathbb{C})$.

The space $\mathcal{N}$ is quite important in representation theory (e.g. remember the talks by Carl and Peng!). If we set $\mathfrak{n} \subset \mathrm{Mat}(n, \mathbb{C})$ the space of strictly upper triangular matrices (the fiber above), then we have

$$\mathcal{N} = G\mathfrak{n} \subset \mathrm{Mat}(n, \mathbb{C}).$$

(The action is the adjoint action.)

Loop groups and affine Grassmaniann

We formally extend $\mathbb{C}$ to $\mathbb{C}[\![z]\!]$ or $\mathbb{C}(\!(z)\!)$ in the matrix entry to obtain

$$G[\![z]\!] := GL(n, \mathbb{C}[\![z]\!]) \quad \text{and} \quad G(\!(z)\!) := GL(n, \mathbb{C}(\!(z)\!)).$$

The theory of Kac-Moody algebra tells us that $G(\!(z)\!)$ admits a central extension that we denote by $\widetilde{G}$. In addition, we have its level one basic representation $L(\Lambda_0)$ that roughly looks like

$$L(\Lambda_0) = \mathbb{C}\mathbf{v}_{\Lambda_0} \oplus \mathfrak{sl}(n, \mathbb{C})z^{-1} \oplus \text{lower degree terms w.r.t. } z$$

Then, the affine Grassmannian looks as

$$\mathrm{Gr} = G(\!(z)\!)/G[\![z]\!] \cong \widetilde{G}[\mathbf{v}_{\Lambda_0}] \subset \mathbb{P}(L(\Lambda_0)).$$

(This is a moral statement and should not be identified with a mathematical statement.)

Lusztig's compactification of the nilpotent cone

We have:

$$\mathrm{Gr} := G(\!(z)\!)/G[\![z]\!] \cong \widetilde{G}[\mathbf{v}_{\Lambda_0}] \subset \mathbb{P}(L(\Lambda_0)).$$

Here, we have

$$\exp(\frac{\mathfrak{n}}{z}) \subset G(\!(z)\!)$$

that lifts to $\widetilde{G}$. If we apply this to $[\mathbf{v}_{\Lambda_0}]$ and additionally apply the $G \subset \widetilde{G}$-action, we find an embedding

$$\mathcal{N} \hookrightarrow \mathrm{Gr} \subset \mathbb{P}(L(\Lambda_0)).$$

This is Lusztig's embedding (1981). A miracle is that, the closure of the image is the closure of a particular $G[\![z]\!]$-orbit of Gr.

Affine Dynkin diagram of $\widetilde{G}$

The affine Lie algebra $\mathrm{Lie}\,\widetilde{G}$ is essentially of type $\mathsf{A}^{(1)}_{n-1}$, whose Dynkin diagram is of shape:

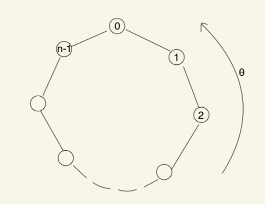

Figure: Dynkin diagram of type $\mathsf{A}^{(1)}_{n-1}$ and the diagram automorphism θ

In particular, θ acts on $\widetilde{G}$ and its set of weights. Therefore, we have

$$K_i := \theta^{n-i}(G[\![z]\!]) \subset \widetilde{G} \quad \text{that have a character} \quad \Lambda_i := \theta^{n-i}\Lambda_0.$$

We have a $\widetilde{G}$-representation $L(\Lambda_i) := (\theta^{n-i})^* L(\Lambda_0)$.

Construction of a variety X_Ψ when $T^*_\Psi X = T^*X$ ... step I

We set $[\mathbf{v}_{\Lambda_i}] = [(\theta^{n-i})^* \mathbf{v}_{\Lambda_0}] \in \mathbb{P}(L(\Lambda_i))$.

We start with $[\mathbf{v}_{\Lambda_0}] \in \mathbb{P}(L(\Lambda_0))$. We have

$$\mathbb{P}^{n-1} \cong K_{n-1}[\mathbf{v}_{\Lambda_0}] \subset \mathbb{P}(L(\Lambda_0))$$

by inspection. Now we consider

$$[\mathbf{v}_{\Lambda_{n-1}}] \times K_{n-1}[\mathbf{v}_{\Lambda_0}] \subset \mathbb{P}(L(\Lambda_{n-1})) \times \mathbb{P}(L(\Lambda_0)).$$

Apply K_{n-2} to this. We can check

$$\mathrm{Stab}_{K_{n-2}}(\mathbf{v}_{\Lambda_{n-1}}) \circlearrowright K_{n-1}[\mathbf{v}_{\Lambda_0}] \cong \mathbb{P}^{n-1}.$$

It follows that

$$K_{n-2}([\mathbf{v}_{\Lambda_{n-1}}] \times K_{n-1}[\mathbf{v}_{\Lambda_0}]) \subset \mathbb{P}(L(\Lambda_{n-1})) \times \mathbb{P}(L(\Lambda_0))$$

defines a $\mathbb{P}^{n-1}$-fibration over $\mathbb{P}^{n-1}$ through the projection to $\mathbb{P}(L(\Lambda_{n-1}))$.

Construction of a variety X_Ψ when $T^*_\Psi X = T^*X$ ... step II

In fact, we can continue this by lowering the index by one for each time (alternatively rotate one step by θ) since

$$\mathrm{Stab}_{K_{n-k}}(\mathbf{v}_{\Lambda_{n-k}}) \circlearrowright \text{previous output} = K_{n-k+1}([\mathbf{v}_{\Lambda_{n-k+2}}] \times Y),$$

where Y is the variety constructed two steps before.

This yields the n-times ($=$ number of Λ_i's) repeated application of $\mathbb{P}^{n-1}$-bundle, that we denote by X_Ψ. In particular, we have

$$\dim X_\Psi = n(n-1) = \dim T^*X.$$

Examining Lustig's construction, the projection to $\mathbb{P}(L(\Lambda_0))$ yields

$$X_\Psi \to \overline{\mathcal{N}} \subset \mathrm{Gr}.$$

The variety X_Ψ when $T^*_\Psi X = T^* X$ ... part I

By construction, we have

$$X_\Psi \hookrightarrow \prod_{i=0}^{n-1} \mathbb{P}(L(\Lambda_i)).$$

Note that the top z-grading part of $L(\Lambda_i)$ is trivial $(i = 0)$ or fundamental representation of G $(i \neq 0)$. Since we have

$$\{[\mathbf{v}_{\Lambda_i}]\}_i \in X_\Psi$$

by construction, we find $X \cong G\{[\mathbf{v}_{\Lambda_i}]\}_i \subset X_\Psi$. Since we already have $\mathfrak{n} \subset \mathbb{P}(L(\Lambda_0))$, we conclude that

$$T^* X \subset X_\Psi,$$

with the natural $\mathbb{C}^\times$-action reverted. It must be dense by the dimension counting.

The variety X_Ψ when $T^*_\Psi X = T^* X$ ... part II

The embedding

$$X_\Psi \hookrightarrow \prod_{i=0}^{n-1} \mathbb{P}(L(\Lambda_i))$$

defines a line bundle $\mathcal{O}_{X_\Psi}(\varpi_i)$ $(0 \le i < n)$ by the pullback of the corresponding $\mathcal{O}(1)$.

Theorem

The line bundle $\mathcal{O}_{X_\Psi}(\varpi_i)$ restricts to $\widetilde{L}_\lambda$ for

$$\lambda = \begin{cases} (1, 1, \ldots, 1) & (i = 0) \\ (\overbrace{1, 1, \ldots, 1}^{i}, 0, \ldots, 0) & (i \neq 0) \end{cases}$$

through the embedding $T^*_\Psi X \subset X_\Psi$. In addition, the open subset $T^*_\Psi X \subset X_\Psi$ is defined as $\mathbf{v}^*_{\Lambda_0} \neq 0$ (this must be affine embedding).

The case of general Ψ ... properties

We define

$$X_\Psi := \overline{T^*_\Psi X} \subset \overline{T^*X} = X_{\Psi_+},$$

where Ψ_+ is the maximal Dyck path that yields $T^*_{\Psi_+}X = T^*X$.

- In particular, $X_{\Psi'} \subset X_\Psi$ if we have $T^*_{\Psi'}X \subset T^*_\Psi X$, that is equivalent to say that the Dyck path Ψ' is always below Ψ (before upsidedown);
- This is a consequence of the original definition;
- Our original construction of X_Ψ makes it possible to see it is a successive $\mathbb{P}^\bullet$-bundle (and hence is smooth projective);
- Though there is no logical dependence, some intermediate steps of the proof (not exhibited here) employs the contents of the talk I had originally planned to give.

The case of general Ψ ... statement

We define

$$X_\Psi := \overline{T^*_\Psi X} \subset \overline{T^* X} = X_{\Psi_+},$$

where Ψ_+ is the maximal Dyck path that yields $T^*_{\Psi_+} X = T^* X$.

For a partition $\lambda \in \mathbb{Z}^n$, we set the coefficient of ϖ_i as $(\lambda_i - \lambda_{i+1})$ (λ_n for $i = 0$) to obtain a line bundle $\mathcal{O}_{X_{\Psi_+}}(\lambda)$ on X_{Ψ_+}.

Theorem (Geometric realization of Catalan polynomials)

For a Dyck path Ψ of size n and a partition λ, we have a line bundle $\mathcal{O}_{X_\Psi}(\lambda)$ obtained as the restriction of $\mathcal{O}_{X_{\Psi_+}}(\lambda)$ such that:

1. *We have a surjective restriction map*

$$H^0(X_{\Psi_+}, \mathcal{O}_{X_{\Psi_+}}(\lambda)) \longrightarrow\!\!\!\!\!\rightarrow H^0(X_\Psi, \mathcal{O}_{X_\Psi}(\lambda));$$

2. $\mathrm{char}_q H^0(X_\Psi, \mathcal{O}_{X_\Psi}(\lambda)) = HL^\Psi_\lambda$;
3. $H^{>0}(X_\Psi, \mathcal{O}_{X_\Psi}(\lambda)) = 0$.

The Chen-Haiman conjecture

Recall that $H^{>0}(X_\Psi, \mathcal{O}_{X_\Psi}(\lambda)) = 0$ when λ is a partition.

Corollary (Chen-Haiman's vanishing conjecture)

$$H^{>0}(T^*_\Psi X, \widetilde{L}_\lambda) = 0 \quad \textit{when } \lambda \textit{ is a partition.}$$

We have

$$H^{>0}(T^*_\Psi X, \widetilde{L}_\lambda) = \varinjlim_m H^{>0}(X_\Psi, \mathcal{O}_{X_\Psi}(\lambda + m\varpi_0)) \otimes \mathbb{C}_{m \det}.$$

Here we have

$$j_* \mathcal{O}_{T^*_\Psi X} = \bigcup_m \mathcal{O}_{X_\Psi}(\lambda + m\varpi_0)) \boxtimes \mathbb{C}_{m \det} \quad \text{and} \quad \mathbb{R}^{>0} j_* \mathcal{O}_{T^*_\Psi X} = 0$$

for $j : T^*_\Psi X \subset X_\Psi$ since j is an affine embedding.

NB The Chen-Haiman vanishing conjecture is previously shown to hold when $T^*_\Psi X = X$ (BWB), $T^* X$ (Broer), sufficiently dominant weights (Panyshev), and other special cases.

A consequence

The map

$$\mu : T^*X \to \mathcal{N}$$

sends $T^*_\Psi X$ to a G-invariant closed subset of $\mathcal{N}$, i.e. the nilpotent orbit closure. This yields a surjection

$$\{\text{Dyck paths of size } n\} \to \{\text{partitions of } n\},$$

and we find that

$$HL^{\Psi}_{(m,m,\ldots,m)} = HL^{\Psi'}_{(m,m,\ldots,m)} \qquad m \geq 0$$

whenever Ψ and Ψ' have a common image. For us, this is a consequence of

$$\Gamma(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}) = \mathbb{C}[\mu(T^*_\Psi X)] \equiv \mathbb{C}[\mu(T^*_{\Psi'} X)] = \Gamma(T^*_{\Psi'} X, \mathcal{O}_{T^*_{\Psi'} X})$$

This is the discussion in the paper of Shimozono-Weyman, that they have casted out their conjectures little modestly.