Hecke algebras and categorification

Joint work with Chris Bowman.

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- H_n^κ is a deformation of $\mathbb{F}((\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n)$.
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- So how do these cellular structures arise, and what is the structure of $\mathsf{S}_\theta^\lambda$?

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$\mathbb{F}$ a field, $e \in \{2, 3, \dots\} \cup \{\infty\}$, $I := \mathbb{Z}/e\mathbb{Z}$ (or $I := \mathbb{Z}$ if $e = \infty$). For $\kappa \in I^\ell$, the **cyclotomic Khovanov–Lauda–Rouquier algebra** R_n^κ is the unital, associative $\mathbb{F}$-algebra with generating set

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$$e(i)e(j) = \delta_{i,j} e(i); \qquad \sum_{i \in I^n} e(i) = 1;$$

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$$y_r y_s = y_s y_r;$$

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y_r \psi_r e(i) &= (\psi_r y_{r+1} - \delta_{i_r,i_{r+1}})e(i);\\
y_{r+1}\psi_r e(i) &= (\psi_r y_r + \delta_{i_r,i_{r+1}})e(i);
\end{aligned}$$

$$\psi_r^2 e(i) = \begin{cases} 0 & i_r = i_{r+1}, \\ e(i) & i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e(i) & i_r = i_{r+1}+1, \\ (y_r - y_{r+1})e(i) & i_r = i_{r+1}-1, \\ (y_{r+1} - y_r)(y_r - y_{r+1})e(i) & i_r = i_{r+1}+1, e = 2; \end{cases}$$

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$$y_1^{\langle\Lambda_\kappa,\alpha_{i_1}\rangle} e(i) = 0;$$

for all admissible r, s, i, j.

Cylotomic KLR algebras and Hecke algebras

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Fact

R_n^κ is $\mathbb{Z}$-graded by setting

$$\deg(e(i)) = 0; \quad \deg(y_r) = 2;$$

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Theorem (Brundan–Kleshchev, '09)

Suppose $e = p$ or $p \nmid e$. Then R_n^κ is isomorphic to the cyclotomic Hecke algebra H_n^κ.

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Now that we may study the *graded* representation theory of the former, we would like a *graded* quasi-hereditary cover of R_n^κ.

In fact, Webster constructed a whole family of graded quasi-hereditary covers of R_n^κ, indexed by an extra parameter, $\theta \in \mathbb{Z}^\ell$.

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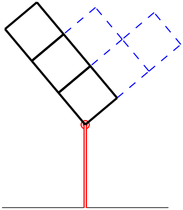

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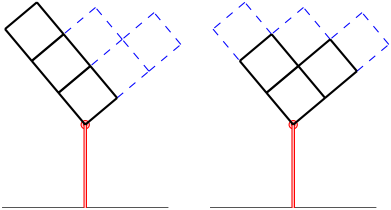

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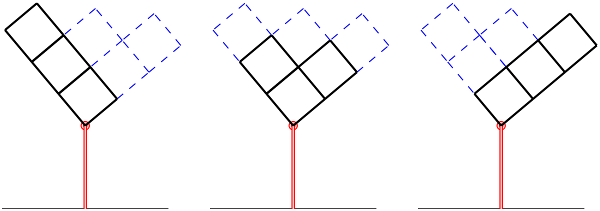

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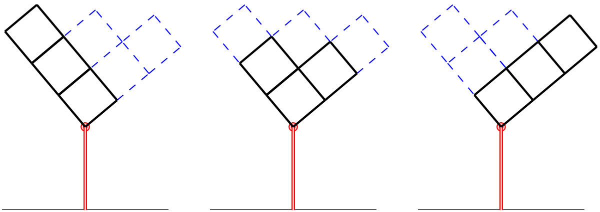

We have an ordering from left-to-right:

$$(3) \trianglerighteq_\theta (2,1) \trianglerighteq_\theta (1^3).$$

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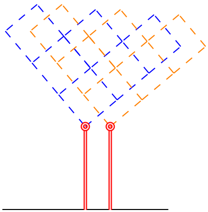

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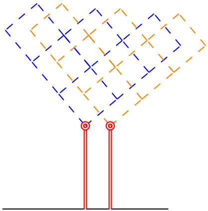

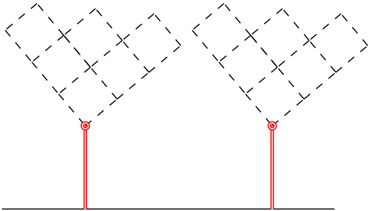

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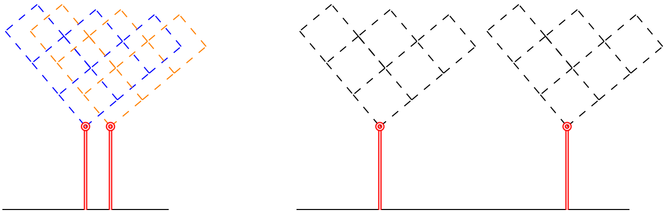

Any θ weighting corresponds to a θ-dominance ordering on $\mathscr{P}_n^\ell$ as follows.

Definition

We write $\lambda \trianglelefteq_\theta \mu$ if for any $x \in \mathbb{R}$ the number of boxes in $[\lambda]_\theta$ to the left of x is less than or equal to the number of points in $[\mu]_\theta$ to the left of x.

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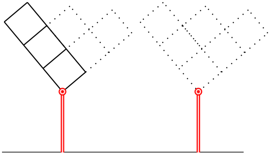

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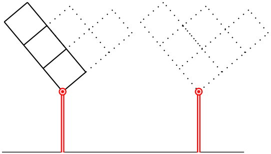

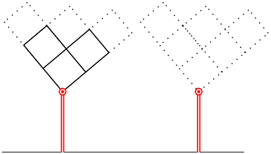

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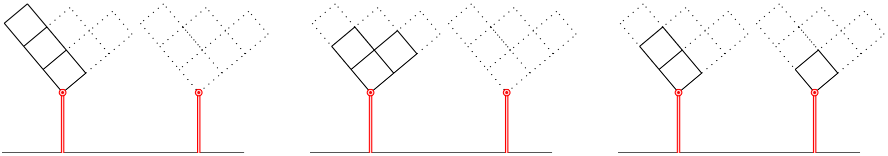

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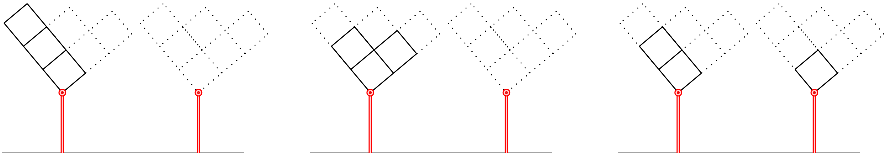

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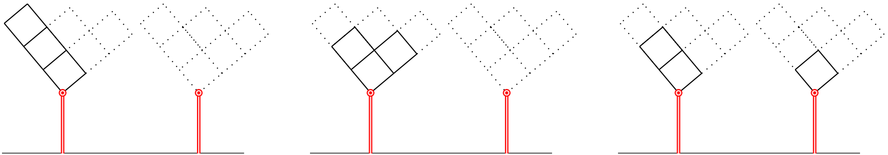

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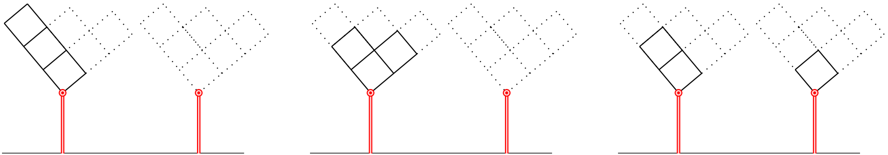

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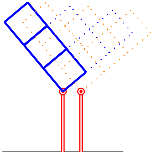

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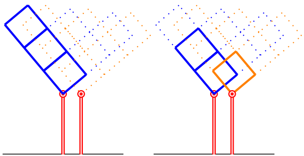

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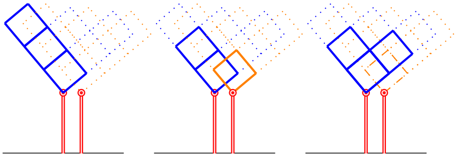

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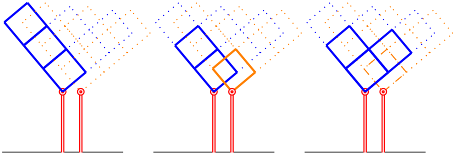

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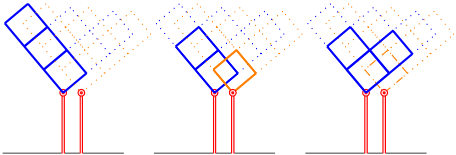

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- This dominance ordering is due to Foda, Leclerc, Okado, Thibon, Welsh, which is why we call such weightings 'FLOTW'.

The diagrammatic Cherednik algebra

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- The simple modules surviving under this idempotent truncation are labelled by Θ.

- For any weighting $\theta \in \mathbb{Z}^\ell$, the diagrammatic Cherednik algebra $A(n, \theta, \kappa)$ is a graded quasi-hereditary cover of the cyclotomic KLR algebra, and in particular

$$[\Delta(\lambda) : L(\mu)] = [\mathsf{S}_\theta^\lambda : D_\theta^\mu]$$

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- It is in fact the truncation from $A(n, \theta, \kappa)$ that gives rise to the corresponding cellular structure on R_n^κ (and H_n^κ).

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- These graded decomposition numbers depend only on the 'relative configurations' of addable and removable r-nodes, not on n or e.

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- We hence deduce that the decomposition numbers (and certain higher extension groups) for $A(n, \theta, \kappa)$ (and R_n^κ) are preserved.

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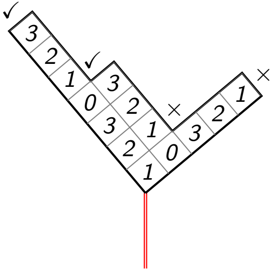

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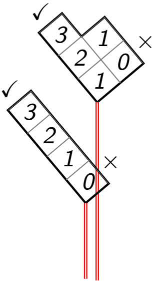

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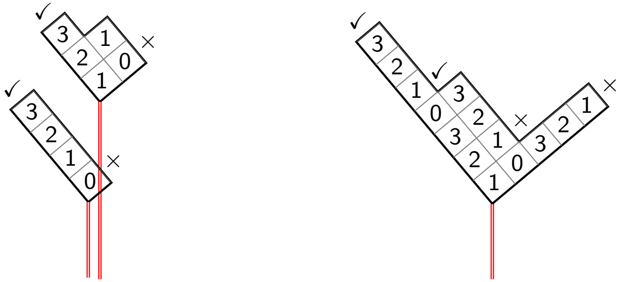

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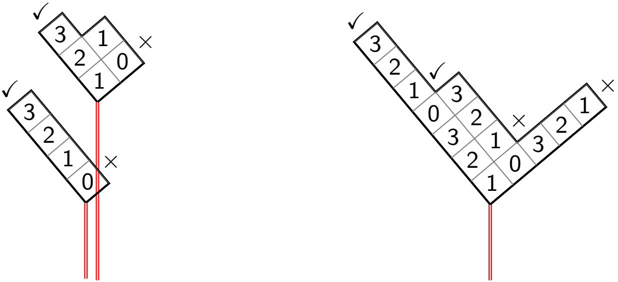

In particular, it allows us to reduce many situations to Tan and Teo's level 1 result, and deduce in the above example that

$$d_{\bar{\lambda}\bar{\mu}} = d_{\lambda\mu} = v^4.$$

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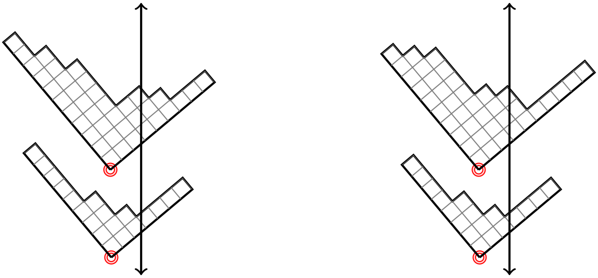

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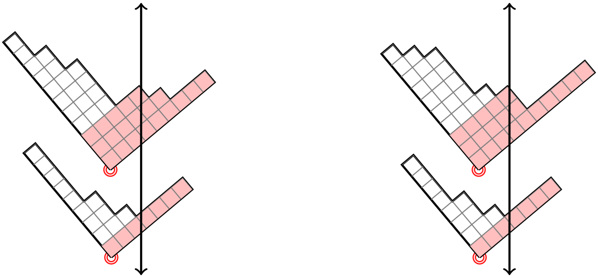

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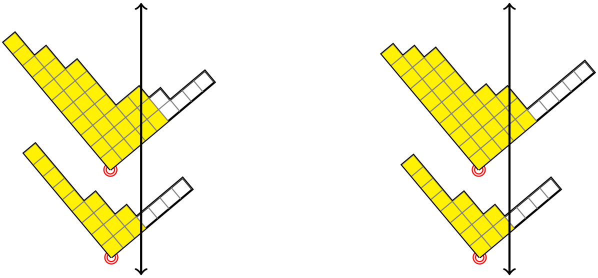
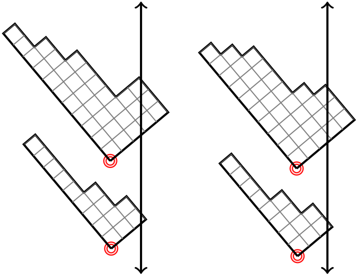


Then we are able to cut λ and μ into some 'left-hand pieces' and 'right-hand pieces', which have some nodes in common.



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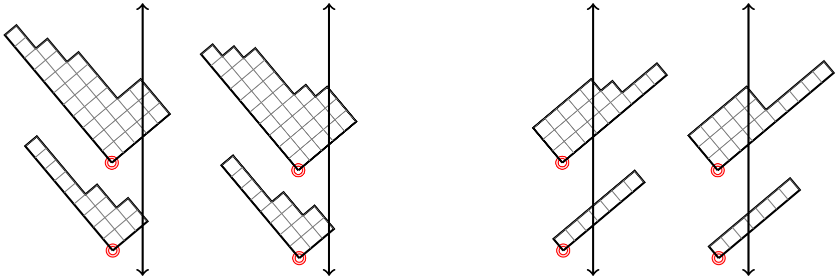

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and the (graded) higher extension groups $Ext^k_{A(n,\theta,\kappa)}(\Delta(\lambda), \Delta(\mu))$ can be decomposed as

$$\bigoplus_{i+j=k} Ext^i_{A(n_L,\theta,\kappa)}(\Delta(\lambda^L), \Delta(\mu^L)) \otimes Ext^j_{A(n_R,\theta,\kappa)}(\Delta(\lambda^R), \Delta(\mu^R)),$$

where $n_L = |\lambda^L| = |\mu^L|$ and $n_R = |\lambda^R| = |\mu^R|$.

Example

*$e = 3$, $\kappa = (0, 1)$, $\lambda = ((5^2, 4^2, 3, 2, 1), (9, 6, 4^2, 3, 2^3, 1))$,
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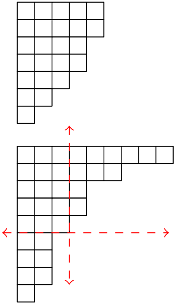
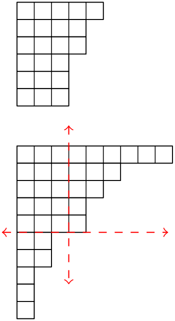
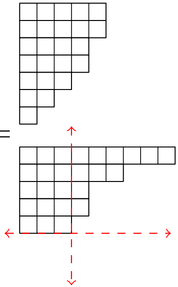
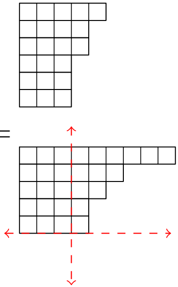
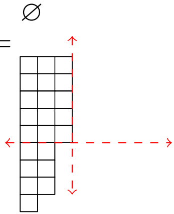
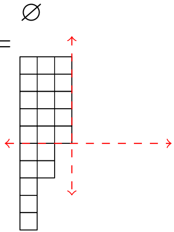
$[\lambda] =$  $[\mu] =$ 

$[\lambda_{\mathrm{T}}] =$ 

$[\mu_{\mathrm{T}}] =$ 

$[\lambda_{\mathrm{B}}] =$ 

$[\mu_{\mathrm{B}}] =$ 

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Thus, we have $d_{\lambda\mu} = v^{12} + 2v^{10} + 2v^8 + v^6$.