Path morphisms in Reduced expressions graphs

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Outline

Involved elements

Element in a Coxeter Group	$\rightarrow$	*Graph*	$\rightarrow$	*Functions*	$\rightarrow$	*Paths in a graph*
Symmetric group S_n		*Defined by reduced expressions*		*Between Bott – Samelson bimodules*		*Defining composition of functions*

Our goal

We want to establish criteria to determine when a pair of paths, starting at the same point and ending at the same point, define the same morphism (naturally, without involving the entire calculus itself).

Motivation

The motivation to learn more about this problem comes from a technique used to explicitly describe an idempotent which picks out a summand inside a Bott-Samelson bimodule.

Coxeter Group

For $n \in \mathbb{N}$, let (W, S) be the Coxeter system with $W = S_n$ the symmetric group on $\{1, \ldots, n\}$, and generator $S = \{s_i \mid i = 1, 2, \ldots, n-1\}$ where each s_i is the transposition $(i\; i+1)$.

When no confusion is possible, we simplify notation writing ijk in place of $s_i s_j s_k$.

Reduced Expressions Graph

The *reduced expressions graph* of an element $w \in W$, usually abreviated *rex graph*, is the graph defined as follows. Its vertices are the reduced expressions of w, with an edge between two reduced expressions if they differ by a single braid relation.

Reduced expressions graph - Braid relations.

These relations are

$$s_i \; s_{i+1} \; s_i = s_{i+1} \; s_i \; s_{i+1}$$

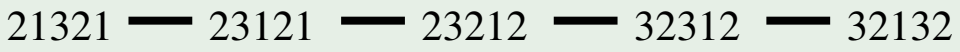
for all $i \in \{1, 2 \ldots, n-2\}$, and

$$s_i \; s_j = s_j \; s_i$$

when $|i - j| \geq 2$. We call the edges determined by the former identity *adjacent edges*, and those determined by the latter, *distant edges*.

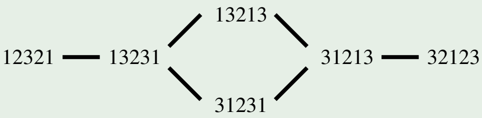
Example

Reduced expressions graph of 21321.



Example

Reduced expressions graph of 12321.



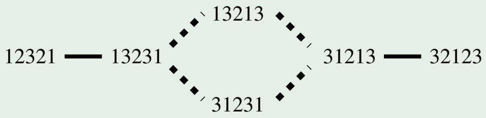
Expanded expressions graph

Definition

Given a rex graph of $w \in W$, we can draw the distant edges with dashed lines. With this convention, we name this colored graph the *expanded expressions graph of* w.

Example

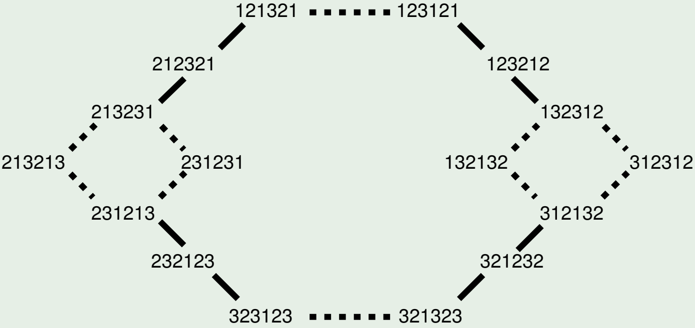
The expanded expressions graph of 12321.



Zamolodchikov cycle

Example

The expanded expressions graph of $w_{0,4}$.



Bott-Samelson Bimodules

Let R be the polynomial ring over $\mathbb{R}$ in n variables $x_1, \ldots, x_n$, together with an action of W where s_i permutes the variables x_i and x_{i+1}.

Notation

For s in S, we denote by R^s the subring of R consisting of polynomials invariants under the action of s. Let B_s denote the graded R-bimodule $B_s := R \otimes_{R^s} R(1)$.

Bott-Samelson Bimodules

Definition

The *Bott-Samelson bimodule* related to an expression $\underline{w} = (s_1, s_2 \ldots, s_n)$, and denoted by $B_{\underline{w}}$, is the graded R-bimodule given by the tensor products of bimodules $B_{\underline{w}} = B_{s_1} \otimes_R B_{s_2} \otimes_R \ldots \otimes_R B_{s_n}$.

We may simplify the notation by writing B_i instead of B_{s_i}. For tensor products, we write B_iB_j instead of $B_{s_i} \otimes_R B_{s_j}$.
We write $1^{\otimes}$ for $1 \otimes 1 \otimes \ldots \otimes 1$.

Morphisms f_{sr}

First case: If $|i-j| \geq 2$. The morphism $f_{ij} : B_iB_j \to B_jB_i$ is determined by the formula $f_{ij}(1^{\otimes}) = 1^{\otimes}$, because $1^{\otimes}$ generates B_iB_j as a bimodule.

Second case: The morphism $f_{i(i+1)} : B_iB_{i+1}B_i \to B_{i+1}B_iB_{i+1}$ is determined by the formulae $f_{i(i+1)}(1^{\otimes}) = 1^{\otimes}$ and

$$f_{i(i+1)}(1 \otimes x_i \otimes 1 \otimes 1) = (x_i + x_{i+1}) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2}.$$

Third case: The morphism $f_{i(i-1)} : B_iB_{i-1}B_i \to B_{i-1}B_iB_{i-1}$ is determined by the formulae $f_{i(i-1)}(1^{\otimes}) = 1^{\otimes}$ and

$$f_{i(i-1)}(1 \otimes x_{i+1} \otimes 1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes (x_i + x_{i+1}) - x_{i-1} \otimes 1 \otimes 1 \otimes 1.$$

Example

In S_4, the expressions 212321, and 213231 are reduced expressions of the same element, and they differ by the braid relation $232 = 323$. The aforementioned morphism from 212321 to 213231 has the following form.

$$Id^2 \otimes f_{23} \otimes Id : B_2 \otimes B_1 \otimes (B_2 \otimes B_3 \otimes B_2) \otimes B_1 \to B_2 \otimes B_1 \otimes (B_3 \otimes B_2 \otimes B_3) \otimes B_1 .$$

Path morphisms

For each path p in the rex graph $Rex(w)$ it is possible to associate a morphism $f(p)$ between the Bott-Samelson bimodules B_{p_i} and B_{p_f}(through the corresponding compositions).

We call $f(p)$ a *path morphism*.

Forking Path Conjecture

Conjecture

Let $x \in S_n$, let p, q be two complete paths with the same starting points and the same ending points in the reduced expressions graph of x. The morphisms induced by these paths are equal.

Counterexample

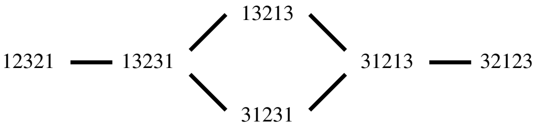
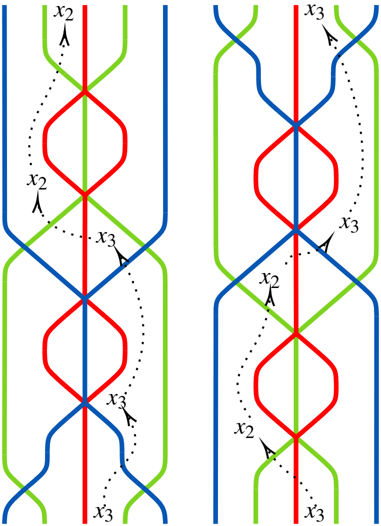


Figure: Reduced expressions graph of 12321

Let us consider the element $x := 1 \otimes_{s_1} 1 \otimes_{s_3} 1 \otimes_{s_2} x_3 \otimes_{s_3} 1 \otimes_{s_1} 1$ in the Bott-Samelson bimodule $B_1 B_3 B_2 B_3 B_1$.

Counterexample



Conflated expressions graph

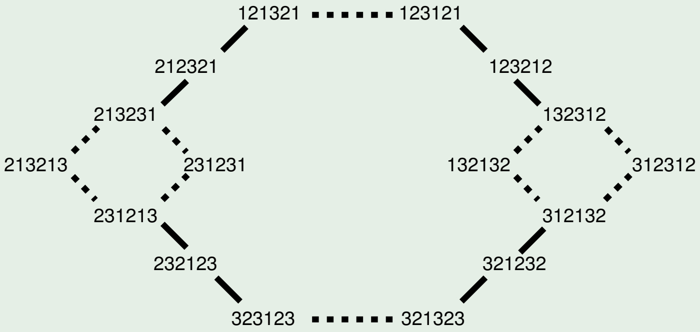
Definition

The *conflated expressions graph* is the quotient of the reduced expressions graph by all its distant edges; that is, we identify any two vertices connected by a distant edge, and remove the distant edge.

Zamolodchikov cycle

Example

The expanded expressions graph of $w_{0,4}$.



Zamolodchikov cycle

Example

The conflated expressions graph of $w_{0,4}$.

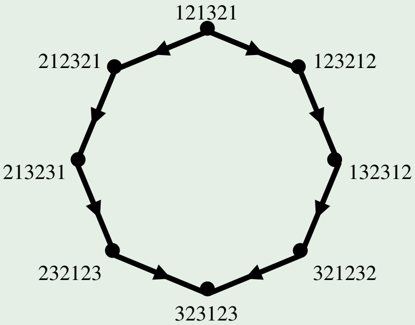
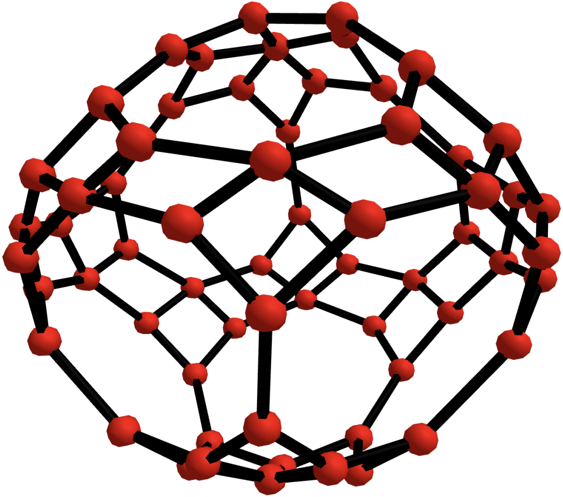
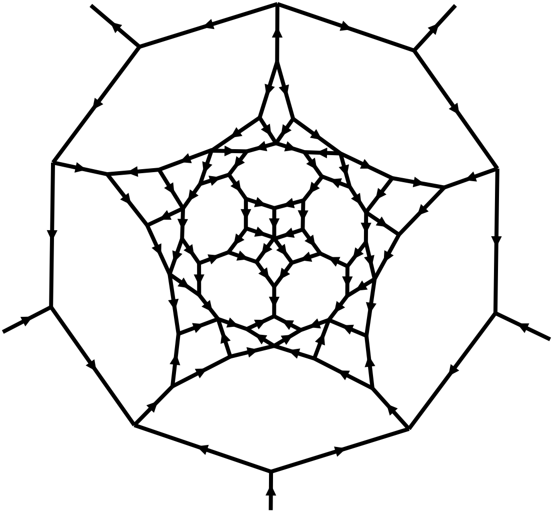


Figure: Manin-Schechtman oriented Zamolodchikov cycle.



Orientations in conflated expressions graphs



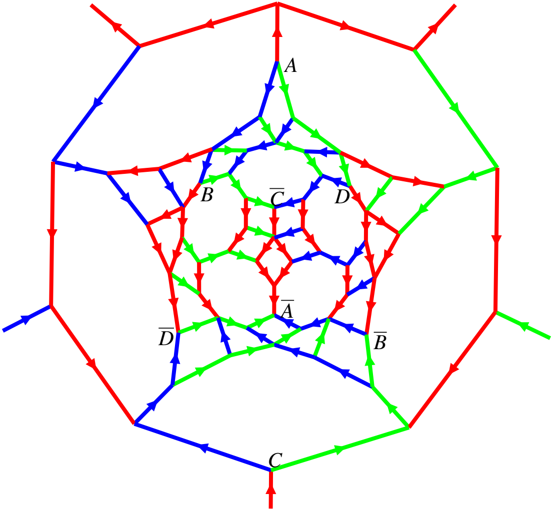


Figure: A=1213214321, $\overline{A}$=4342341234; B=2341213214, $\overline{B}$=1434234123; C=3412132143, $\overline{C}$=2143423412; D=3423121434, $\overline{D}$=4121321432.

Summary

- We did a review of path morphisms in Rex graphs.
- We saw a counterexample for a conjecture.
- We know how to find new “interesting” orientations in Rex graphs

Thank you.

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