Universal construction, one-dimensional cobordisms with defects, and pseudocharacters

Representation Theory of Hecke Algebras and Categorification
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Joint work.

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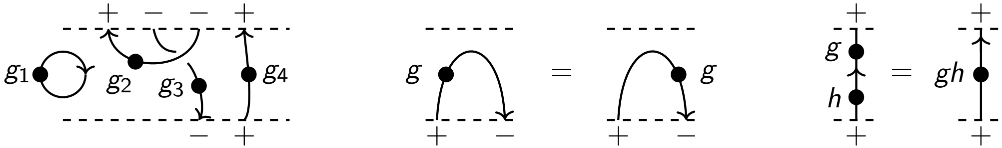
Category $\mathrm{Cob}_{1,G}$ of G-decorated 1-cobordisms.

Illustrate notions of universal construction, a topological theory and a TQFT with an example.

For a group G (or, more generally, a monoid G) consider the category $\mathrm{Cob}_{1,G}$ of oriented 1-cobordisms decorated by dots labelled by elements of G. Objects are oriented 0-manifolds (sign sequences $\varepsilon = (\pm, \ldots, \pm)$).



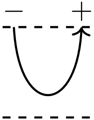
Dots can freely slide along components of a cobordism. Dots g, h can merge into the dot gh (need strand orientation for that).

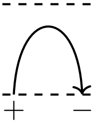
A topological quantum field theory (TQFT) for this category (over $\mathbb{C}$) is a tensor (symmetric monoidal) functor

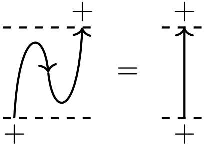
$$F \; : \; \mathrm{Cob}_{1,G} \longrightarrow \mathbb{C}\text{–vect}.$$

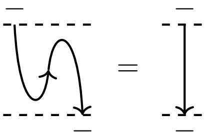
Theorem

A TQFT for $\mathrm{Cob}_{1,G}$ is a finite-dimensional representation of G.

ψ 

ϕ 





Indeed, $V := F(+)$ is a vector space and let $V' = F(-)$. Cup and cap morphisms ψ and ϕ above induce maps

$$F(\psi) \ : \ \mathbb{C} \longrightarrow V' \otimes V, \qquad F(\phi) \ : \ V \otimes V' \longrightarrow \mathbb{C}$$

and the isotopy relations imply isomorphisms $V' \cong V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ such that ψ and ϕ are the standard diagonal and evaluation maps:

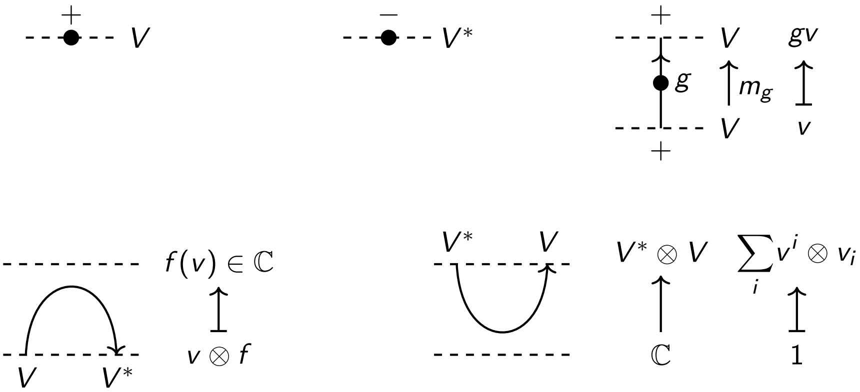
$$\psi(1) = \sum_{i=1}^{n} v^i \otimes v_i, \qquad \phi(v_i \otimes v^j) = \delta_{i,j}.$$

Here $n = \dim(V)$, $(v_1, \ldots, v_n)$ is a basis of V, and $(v^1, \ldots, v^n)$ is the dual basis of V^*.

Thus, to $+$ and $-$ endpoints we assign vector space V and its dual V^*, respectively.

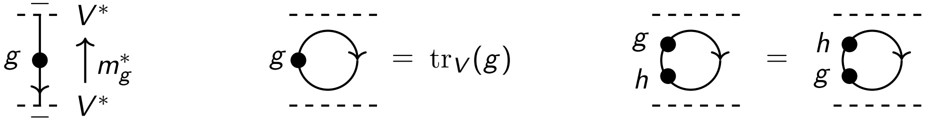
A defect labelled $g \in G$ on an upward-oriented line defines a map $m_g : V \longrightarrow V$.

The maps satisfy the composition law $m_{gh} = m_g m_h$ and arrange into a (finite-dimensional) representation V of G.



Sliding a dot along a cup or a cap shows that to a dot labelled g on a downward-oriented arrow, TQFT F assigns the dual map $m_g^* : V^* \longrightarrow V^*$.

A g-dotted circle evaluates to $\mathrm{tr}_V(g) = \chi_V(g)$, the character of $g \in G$ on V.



Conjugation invariance of characters $\chi_V(gh) = \chi_V(hg)$ has a graphical interpretation given by sliding a dot around the circle.

Topological theories.

In contrast to a *TQFT*, in a *topological theory* we are only given *evaluations of closed manifolds*. To a g-decorated circle, assign a number $f(g) \in \mathbb{C}$.

Isotopy property of moving a dot around the circle implies that $f(gh) = f(hg)$ for $g, h \in G$, so f is a function (constant) on conjugacy classes

$$f \ : \ \mathcal{O}_G \longrightarrow \mathbb{C}$$

$\Longrightarrow$ f extends linearly to the group algebra of G, i.e.,

$$f : \mathbb{C}G \longrightarrow \mathbb{C}, \qquad f\left(\sum_i \lambda_i g_i\right) = \sum_i \lambda_i f(g_i).$$

Given such f, one can build state spaces $A(\varepsilon)$ for oriented 0-manifolds ε (sign sequences) by looking at all G-decorated one manifolds M_i that bound ε and imposing the relation

$$\sum_{i=1}^{k} \lambda_i [M_i] = 0$$

iff for any M with $\partial M = \varepsilon$,

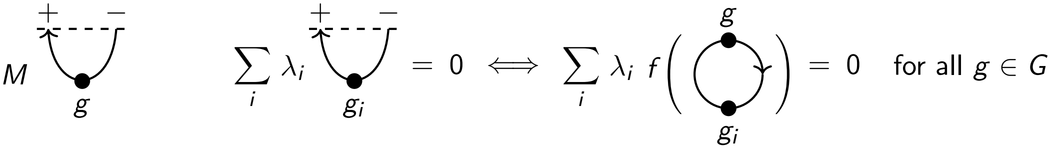
$$\sum_{i=1}^{k} \lambda_i f(\overline{M}\, M_i) = 0.$$

$A(+-)$.

If $\partial M = \varepsilon$, sequence ε has equal number of $+$'s and $-$'s (a *balanced* sequence). In particular, the state space $A(\varepsilon) = 0$ if $|\varepsilon| := |\varepsilon|_+ - |\varepsilon|_- \neq 0$ (if ε is unbalanced; no manifolds M with $\partial M = \varepsilon$ exist).

Example: $\varepsilon = +-$. Manifold M with $\partial M = +-$ is a g-labelled arc $\cup_g$. A linear combination $\sum_i \lambda_i [\cup_{g_i}] = 0$ iff

$$\sum_i \lambda_i f(gg_i) = 0 \quad \forall g \in G.$$



For longer balanced sequences, such as $+--++-$, there are many ways ($k!$ ways, where $k = |\varepsilon|_+ = |\varepsilon|_-$) to pair up $+$'s and $-$'s by arcs and then decorate arcs by elements of G, giving more complicated state spaces $A(\varepsilon)$.

A cobordism M in $\mathrm{Cob}_{1,G}$ from ε' to ε induces a map of state spaces

$$A(M) \ : \ A(\varepsilon') \longrightarrow A(\varepsilon), \qquad [M'] \longrightarrow [MM'],$$

given by composing M' whose $\partial(M') = \varepsilon'$ with M to get the cobordism MM' with $\partial(MM') = \varepsilon$.

These maps fit together to give a functor $A_f : \mathrm{Cob}_{1,G} \longrightarrow \mathbb{C}-\mathrm{vect}$ (recall that we started with a function $f : \mathcal{O}_G \longrightarrow \mathbb{C}$ on conjugacy classes). This functor is a *lax TQFT*, in the sense that natural maps

$$A(\varepsilon) \otimes A(\varepsilon') \longrightarrow A(\varepsilon\varepsilon')$$

are only *inclusions*, not isomorphisms.

Lifting problem.

For a TQFT F: $\quad F(\varepsilon\varepsilon') \cong F(\varepsilon) \otimes F(\varepsilon')$.

We say that topological theory A_f lifts to a TQFT F if there are maps of state spaces $\gamma(\varepsilon) : A(\varepsilon) \longrightarrow F(\varepsilon)$ over all sequences ε that intertwine maps $A(M)$ with maps $F(M)$ over all morphisms M in $\mathrm{Cob}_{1,G}$.

In particular, γ respects evaluation of closed cobordisms (g-decorated circles). Maps $\gamma(\varepsilon)$ above are necessarily inclusions, since $A(\varepsilon)$ is defined via a nondegenerate pairing with $A(\varepsilon^*)$ and γ respects the pairing.

Lifting problem: Given a topological theory A_f, does it lift (embed) into some TQFT F?

This question can be posed for various categories of (decorated) cobordisms. There are immediate obstructions to such a lifting.

Finite groups example.

In our case, the category of cobordisms is $\mathrm{Cob}_{1,G}$. If topological theory A_f associated to function f is liftable to a TQFT F given by a representation V, then a circle decorated by $1 \in G$ evaluates to $\dim V$, giving an example of obstruction (no lifting if $f(1) \in \mathbb{C} \setminus \mathbb{N}$).

Example: Suppose G is finite. Then any $\mathbb{C}$-valued function f on conjugacy classes is a unique $\mathbb{C}$-linear combination of characters of irreducible representations V_i:

$$f(g) \;=\; \sum_{i=1}^{m} \lambda_i \chi_i(g), \quad \chi_i(g) = \mathrm{tr}_{V_i}(g), \quad \lambda_i \in \mathbb{C}.$$

A lifting to a TQFT exists iff function f is a *character*, that is, $\lambda_i \in \mathbb{N} = \{0, 1, \ldots\}$ for all i. Then $V = \bigoplus_{i=1}^{k} V_i^{\lambda_i}$ gives a TQFT which lifts function f.

Any G and number theory.

Suppose G is arbitrary. It turns out that the lifting property for functions f on G has been investigated by number theorists starting over 30 years ago (A.Wiles 1988, R.Taylor 1991, R.Rouquier, L.Nissen, J.Bellaiche, G.Chenevier and others) without ever mentioning TQFTs and topological theories.

They are interested primarily in the case when G is a large Galois group, such as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, but their results often hold for any G. In number theory, one sometimes has a function on G and wants to prove that it is the character of a representation of G (a Galois representation).

In number theory, function f on G takes values in more general fields or in local rings, but here we specialize to $\mathbb{C}$.

What are additional properties on $f : G \longrightarrow \mathbb{C}$, besides conjugacy-invariance, for it to be the character of a representation?

Exterior powers.

Exterior power $\Lambda^n V$ is subrepresentation of $V^{\otimes n}$, with surjection and inclusion maps

$$V^{\otimes n} \xrightarrow{p} \Lambda^n V \xrightarrow{\iota} V^{\otimes n}, \quad p \circ \iota = \mathrm{id}_{\Lambda^n V}$$

The idempotent $e_n^- := \iota p$ of projection onto $\Lambda^n V$ is the full antisymmetrizer,

$$e_n^- \;=\; \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sigma,$$

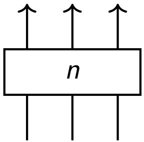
the sum over all permutations σ in S_n.

Key observation: If $n > d := \dim V$, then $\Lambda^n V = 0$. In particular, the composition $h \circ e_n^- = 0$ for any linear operator h on $V^{\otimes n}$, and its trace $\mathrm{tr}(h \circ e_n^-) = 0$ as well.

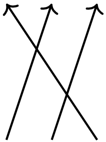
For a linear operator, we choose the product of multiplications by g_i operators, $h = m_{g_1} \otimes \cdots \otimes m_{g_n}$. (Note that any permutation can be swallowed by e_n^-, that is, $\tau e_n^- = (-1)^{\ell(\tau)} e_n^-$, so we keep connecting lines straight.)

Diagrammatics.

To relate to topological theories and TQFTs, we convert to graphical notation.

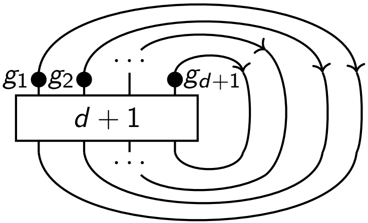


$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\ell(\sigma)}$$



σ

antisymmetrizer e_n^-



$= \quad 0$

closure of e_{d+1}^- is 0
for each $g_1, \ldots, g_{d+1}$ and
closure of e_d^- is not zero
for some $g_1, \ldots, g_d$

pseudocharacter equation

Pseudocharacters.

Algebraically, the pseudocharacter equation is written as

$$\mathrm{tr}_f((g_1 \otimes \cdots \otimes g_{d+1}) \circ e^-_{d+1}) = 0 \quad \text{for each } \ g_1, \ldots, g_{d+1} \in G.$$

What the above means is take $f(\text{closure}((g_1 \otimes \cdots \otimes g_{d+1}) \circ e^-_{d+1})) = 0$ for each $g_1, \ldots, g_{d+1} \in G$.

Definition

A conjugation-invariant function $f : G \longrightarrow \mathbb{C}$ is called a *pseudocharacter* of degree d if any $(g_1, \ldots, g_{d+1})$-closure of the $(d+1)$-antisymmetrizer evaluates to 0 by f, and some closure of the d-antisymmetrizer does not.

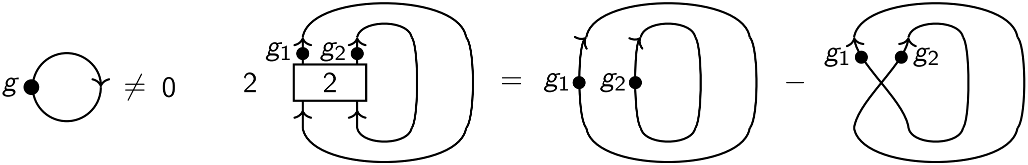
Most functions $f : \mathcal{O}_G \longrightarrow \mathbb{C}$ are not pseudocharacters of any degree.

A character χ_V of a representation V of G of dim d is a pseudocharacter of degree d.

A pseudocharacter f of degree d satisfies $f(1) = d$ (mimicking the equation $\chi_V(1) = \dim V$).

Example: degree 1 pseudocharacters.

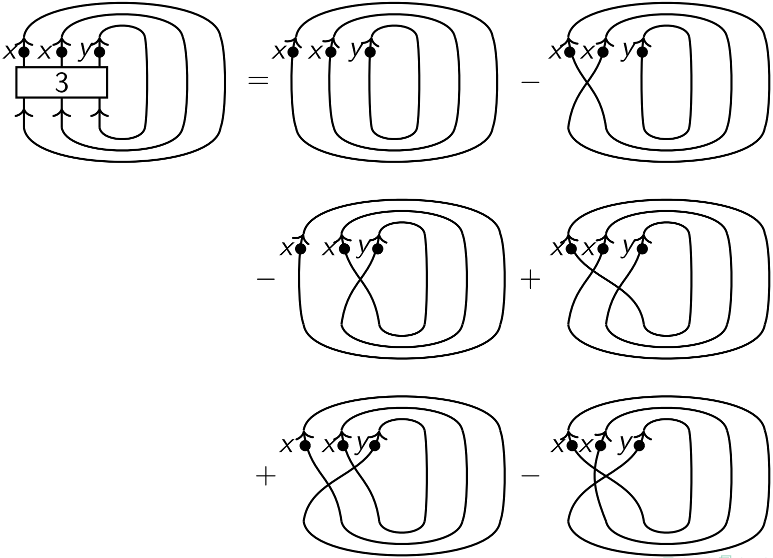
Suppose f is a pseudocharacter of degree one. Then



$$f(g_1)f(g_2) - f(g_1g_2) = 0 \;\Rightarrow\; f(g_1g_2) = f(g_1)f(g_2) \;\; \forall g_1, g_2;$$
$$\exists g \in G: \;\; f(g) \neq 0, \quad f(g1) = f(g)f(1) \;\Rightarrow f(1) = 1 \;\Longrightarrow$$
$$f : G \longrightarrow \mathbb{C}^* \text{ is a homomorphism}$$

We get a 1D representation of G on $\mathbb{C}v$ with $gv = f(g)v$. Hence, any degree 1 pseudocharacter is a character of a one-dimensional representation of G.

Example: degree 2 pseudocharacter computation.



Since the diagram on the previous page is closed, we apply f to obtain:

$$\begin{aligned}
&\mathrm{tr}_f((x \otimes x \otimes y) \circ e_3^-) \\
&\quad = f(x)^2 f(y) - f(x^2)f(y) - f(xy)f(x) + f(x^2y) + f(x^2y) - f(xy)f(x) \\
&\quad = 2f(x^2y) - 2f(xy)f(x) + f(x)^2 f(y) - f(x^2)f(y) \\
&\quad = f((2x^2 - 2f(x)x + f(x)^2 - f(x^2))y) \\
&\quad = f(2\underbrace{\left(x^2 - f(x)x + \frac{1}{2}f(x)^2 - \frac{1}{2}f(x^2)\right)}_{\text{characteristic poly. of a } 2\times 2 \text{ matrix}} y) = 0.
\end{aligned}$$

Illustrates that a special case of the pseudocharacter equation is related to the characteristic polynomial.

See Dotsenko (Prop. 3) to see that $f(1) = 2$ (more generally, $f(1) = d$).

See Dotsenko (Theorem 4) to see that f is a character of a rep of dim 2.

Theorem (R.Taylor, also A.Wiles, R.Rouquier, L.Nissen)

For any group G, if $f : \mathcal{O}_G \longrightarrow \mathbb{C}$ is a pseudocharacter of degree d, then f is a character of a representation $G \longrightarrow \mathrm{GL}(V)$ of dimension d.

We see that in $\mathrm{Cob}_{1,G}$,

$$f \text{ is a character of dimension } d \Longrightarrow f \text{ is a pseudocharacter of degree } d.$$

The theorem above implies

$$f \text{ is a character of dimension } d \Longleftarrow f \text{ is a pseudocharacter of degree } d.$$

Taylor et. al. generalize to some other fields and to local rings.

Case when $d!$ is not invertible in the ground ring is more complicated (A.Wiles for $d = 2$ and G.Chevenier and others for arbitrary d). In this case idempotents e_k^- of projection onto $\Lambda^k V$ are not available, since we cannot divide by $k!$. This should correspond to diagrammatics where one introduces lines of various thickness k up to d to mimic traces of g on exterior powers of V (work in progress).

Given a category $\mathcal{C}$ of cobordisms (for instance, cobordisms in dimension n), a topological theory f is a multiplicative $\mathbb{C}$-valued function on closed cobordisms (cobordisms between empty $(n-1)$-manifolds):

$$f(M_1 \sqcup M_2) = f(M_1)f(M_2), \quad f(\emptyset_n) = 1.$$

We call it a *pseudocharacter* if for any $(n-1)$-manifold N there exists $d = d_N$ such that any closure of the idempotent $e^-_{d+1,N}$ evaluates to 0 under f. The smallest such d is called *the degree of* N *w.r.t.* f.

Here $e^-_{d+1,N}$ is the antisymmetrizer from $N^{\sqcup(d+1)}$ to itself (alternating sum of all permutation cobordisms between $d+1$ copies of N). Any cobordism from $N^{\sqcup(d+1)}$ to itself can be composed with $e^-_{d+1,N}$ to form the closure and then evaluated via f.

2D topological theories with no defects.

Let Cob_2 be the category of oriented two-dimensional cobordisms and f a topological theory for Cob_2. Closed 2-manifolds are disjoint unions of surfaces of genus $0, 1, 2, \ldots$ and f is determined by its values $f_0, f_1, f_2, \ldots$ on these surfaces.

Theorem (M.Khovanov, R.Sazdanovic)

State spaces for f are finite-dimensional iff the generating function

$$Z_f(T) = f_0 + f_1 T + f_2 T^2 + \ldots$$

is a rational function.

Hankel matrix and recurrent relation.

A Hankel matrix is a matrix in which the elements along each anti-diagonal are equal.

$$M = \begin{array}{c c} & \begin{array}{ccccc} v_0 & v_1 & v_2 & v_3 & \dots \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ v_2 \\ \vdots \end{array} & \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \ddots \\ \lambda_1 & \lambda_2 & \lambda_3 & \ddots & \ddots \\ \lambda_2 & \lambda_3 & \ddots & \ddots & \ddots \\ \lambda_3 & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

$A(S^1) := \bigoplus_{i=0}^{\infty} \mathbb{C} v_i / \ker M$, where one obtains the entries in M via a pairing.

An infinite (Hankel) matrix has finite rank $\iff$ there exists k for all $N \gg 0$ such that $\lambda_{N+k} = b_1 \lambda_{N+1} + b_2 \lambda_{N+2} + \ldots + b_{k-1} \lambda_{N+k-1}$ (recurrent relation).

Example of a recurrent relation ($N \geqslant 3$, $k = 3$):

$$\begin{aligned} \lambda_6 &= 17\lambda_5 - 23\lambda_4 \\ \lambda_7 &= 17\lambda_6 - 23\lambda_5 \\ \lambda_8 &= 17\lambda_7 - 23\lambda_6 \\ \vdots &= \qquad \vdots \end{aligned}$$

Therefore, $\sum_{i \geqslant 0} \lambda_i T^i = \dfrac{P(T)}{Q(T)}$, a ratio of two polynomials $\Longleftrightarrow$ we have the above recurrent relations $\Longleftrightarrow$ $A(S^1)$ is finite-dimensional.

Recurrence implies that a surface of high genus g with one boundary (S^1) component is a linear combination of surfaces of lower genera, leading to finite-dimensionality of state spaces.

Theorem (M.S.Im, M.Khovanov, V.Ostrik)

A function f as above is a pseudocharacter for Cob_2 iff it is a character, that is, comes from some two-dimensional TQFT.

A 2D TQFT is given by a commutative Frobenius algebra (B, tr), where $\mathrm{tr} : B \longrightarrow \mathbb{C}$ is a nondegenerate trace. A genus one cobordism with one boundary component defines the handle element $h_B \in B$, and a surface of genus m evaluates to $\mathrm{tr}(h_B^m)$.

We prove the theorem by classifying generating functions for pseudocharacters and checking that all of them are generating functions of 2D TQFTs.

Example: If $B = \mathbb{C}$ is one-dimensional, $\mathrm{tr}(1) = \lambda^{-1}$ for some $\lambda \in \mathbb{C}^*$ and $h_B = \lambda$. Then $f_m = \mathrm{tr}(h_B^m) = \lambda^m \mathrm{tr}(1) = \lambda^{m-1}$ and the generating function of this TQFT is

$$Z_f(T) = \lambda^{-1} + T + \lambda T^2 + \lambda^2 T^3 + \ldots = \frac{\lambda^{-1}}{1 - \lambda T}.$$

Theorem (M.S.Im, M.Khovanov, V.Ostrik)

A function f as above is a pseudocharacter for Cob_2 *iff its generating function has the form*

$$\begin{aligned} Z_f(T) &= \mu + mT + \sum_{i=1}^{s} \frac{m_i\lambda_i^{-1}}{1-\lambda_i T}, \ \mu \in \mathbb{C}, \ m \in \{2,3,\dots\}, \ m_i \in \{1,2,\dots\}, \ \lambda_i \in \mathbb{C}^*, \\ Z_f(T) &= \sum_{i=1}^{s} \frac{m_i\lambda_i^{-1}}{1-\lambda_i T}, \quad m_i \in \{1,2,\dots\}. \end{aligned}$$

Sums that involve λ_i's are characters of one-dimensional Frobenius algebras from the previous example (m_i is the multiplicity), while the term $\mu + mT$ is the character of the non-semisimple Frobenius algebra $\mathbb{C}[x]/(x^m)$ of dimension $m \geqslant 2$ with

$$\mathrm{tr}(1) = \mu, \quad \mathrm{tr}(x^{m-1}) = 1, \quad \mathrm{tr}(x^k) = 0, \ 1 \leqslant k \leqslant m-2, \quad h_B = mx^{m-1}.$$

In the latter TQFT, any surface of genus $\geqslant 2$ evaluates to zero since $h_B^2 = 0$.

Thank you!