Homomorphisms into Specht modules labelled by hooks when $e = 2$

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Hecke algebras

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The representation theory of $\mathcal{H}_n$ is very similar to the modular representation theory of the symmetric group $\mathfrak{S}_n$.

Define the **quantum characteristic** of $\mathcal{H}_n(q)$, $e \geq 2$, to be the smallest integer such that $1 + q + q^2 + \cdots + q^{e-1} = 0$.
Throughout this talk, we fix $e = 2$ (i.e. $q = -1$).

Combinatorics

We have **partitions** $\lambda \vdash n$, **Young diagrams** $[\lambda]$ and **tableaux** $\mathtt{T}$ (in particular, **standard tableaux** of shape λ, denoted $\mathrm{Std}(\lambda)$).

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$$[\lambda] = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 & 1 \\ \cline{1-2} 1 & 0 \\ \cline{1-2} \end{array}.$$

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For $\mathtt{T} \in \operatorname{Std}(\lambda)$, we define the **residue sequence** $\boldsymbol{i}^{\mathtt{T}}$ of $\mathtt{T}$ to be the sequence of residues of nodes containing $1, \ldots, n$ in order.

Cyclotomic KLR algebras

Let $\mathcal{O}$ be a commutative ring with identity.
The cyclotomic **Khovanov–Lauda–Rouquier algebra** of type A, $R_n^{\Lambda_0} = R_n^{\Lambda_0}(\mathcal{O})$, is defined to be the unital $\mathcal{O}$-algebra generated by the elements

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I^n\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\},$$

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Theorem (Brundan–Kleshchev, 2009)

The algebras $R_n^{\Lambda_0}$ and $\mathcal{H}_n$ are isomorphic.

Specht modules

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Relations for z^λ

1. $e(\boldsymbol{i})z^\lambda = \delta_{\boldsymbol{i},\boldsymbol{i}^\lambda} z^\lambda$;
2. $y_r z^\lambda = 0$ for all $1 \leq r \leq n$;
3. $\psi_k z^\lambda = 0$ whenever k and $k+1$ are in the same row of $\mathtt{T}^\lambda$;
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Theorem (Brundan–Kleshchev–Wang, 2011)

Let $\lambda \in \mathscr{P}_n$. Then the universal Specht module $S^\lambda(\mathcal{O})$ for $R_n^{\Lambda^0}(\mathcal{O})$ has homogeneous $\mathcal{O}$-basis

$$\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathsf{Std}(\lambda)\}.$$

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We start with the first relation $e(\boldsymbol{i})f(z^\mu) = \delta_{\boldsymbol{i}^\mu, \boldsymbol{i}} f(z^\mu)$. This tells us that

$$v := f(z^\mu) = \sum_{\substack{\mathtt{T} \in \mathrm{Std}(\lambda) \\ \boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}^\mu}} c_{\mathtt{T}} v_{\mathtt{T}} \in S^\lambda.$$

That is, all $\mathtt{T}$ appearing in v have the same residue sequence as $\mathtt{T}^\mu$.

Difficulties with $e = 2$

1. Our residue patterns only contain 0's and 1's. This is not very helpful!

0	1	2	0	1	2
2					
1					
0					
2					

$e = 3$

0	1	0	1	0	1
1					
0					
1					
0					

$e = 2$

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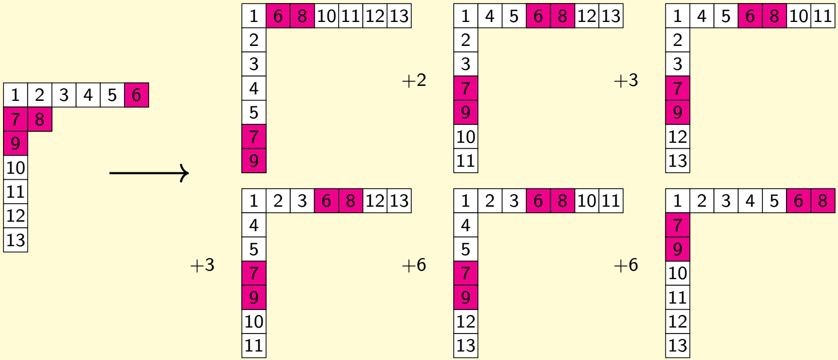

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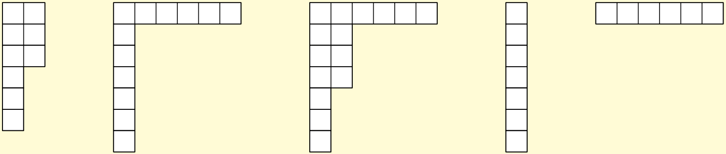

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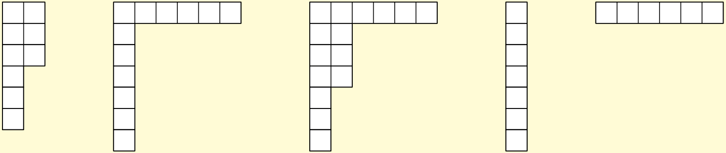

Theorem (H., 2023)

Let $\mu = (c, 2^k, 1^d)$ *and* $\lambda = (a, 1^b)$ *with* c *even. Then* $\dim \mathrm{Hom}_{R_n^{\Lambda_0}}(S^\mu, S^\lambda) \geq 1$ *if and only if:*

- $a \geq k + 2$;
- *if* n *is odd,* $n + a \equiv k + 1 \mod 2$, *or*
- *if* n *even,* $k + 2 \leq a \leq c + k$ *or* $a = c + k + 1 + 2i$ *for* $0 \leq i \leq d/2 - 2$.