Finite dimensional Hecke algebras and beyond

Aim: Results, problems from representation theory of finite-dimensional algebras, specialized to Hecke algebras, focussing on type A.

[Ariki] (Intro of paper dedicated to R. Dipper's retirement, J. Austr. Math. Soc.)

The modular representation theory of Hecke algebras itself is still far from well-understood ...

$\mathcal{H} = H_q(d)$ type A Hecke algebra, over K alg closed field, usually char$(K) = p$. Fix $0 \neq q \in K$ primitive ℓ-th root of 1 (or $q = 1$).

Plan (I) $\mathcal{H}$ viewed as a f-dim algebra.

(II) $\mathcal{H}$ via Schur algebras

(I) Representation type for $\mathcal{H}$

Uno, Geck, E-Nakano, Ariki, Mathas, etal.

finite $\longrightarrow$ Brauer tree algebras, tree a line

Qu same for any Hecke algebra?

tame $\longrightarrow$ Brauer graph algebras

Qu same for any Hecke algebra?

wild (otherwise) $\longrightarrow$???

Other finiteness conditions?

Call a module M Schurian if $\text{End}_A(M) \cong K$. The algebra A is Schurian finite: # Schurian modules (up to iso) is finite.

[Demonet-Jasso-Iyama] $\Leftrightarrow$ τ-tilting finite.

THM [Ariki-Lyle-Speyer] $A =$ block of $\mathcal{H}$. If $\ell \geq 3$ then all blocks of weight $w \geq 2$ are Schurian infinite (any characteristic).

Graded decomposition numbers give part of ext quiver, 'zigzag modules'

[E-Nakano, Doty-EN-Martin] Representation type for $S_q(n, d)$. [Qi Wang] Schurian finitenes for Schur algebras.

Wild case: Stable Auslander Reiten quiver Γ_s

$A =$ wild block, selfinjective, M indecomposable not projective.
Let $\Theta =$ component of Γ_s containing M.

Qu graph structure of Θ?

Tool: Periodic modules.

THM [Happel, Preiser, Ringel, Webb, Okuyama] If there is a periodic module W and $\underline{\mathrm{Hom}}(W, M) \neq 0$ then
(a) $\Theta \cong \mathbb{Z}T$ with T Euclidean or $A_\infty, A_\infty^\infty$ or D_∞.
(b) $A =$ wild block of KG then only A_∞ occurs. (Applications: exclude some Cartan matrices).

AR quiver: directed graph.
Vertices $[M]$ (iso classes of indecomposables). Arrows $[M] \to [N]$ from irreducible maps.

Qu A wild block of $\mathcal{H}$, do we always have such periodic W? Only A_∞?

(a) Analog to group case:
Elementary abelian p-groups $\longrightarrow$ ℓ-parabolic $\mathcal{H}_\rho \subset \mathcal{H}$
(tensor product of $\mathcal{H}(\ell)$'s)

$\Rightarrow$ Works OK if M restricted to $\mathcal{H}_\rho$ is not not projective.

Problem When is M restricted to $\mathcal{H}_\rho$ projective?

(b) Evidence:
THM [S. Schmider] $\ell \geq 3$ and char(K) does not divide $\ell - 1$, then all components for a block of wild type are $\mathbb{Z}A_\infty$.

For KG, support varieties via $H^*(G)$. For $\mathcal{H}$, no Hopf structure but could try $H^*(\mathcal{H}) := \mathrm{Ext}^*_{\mathcal{H}}(K,K)$.

Known when char$(K) = 0$ [Benson-E-Mikaelian] With this, (Nakano-Xiang) introduce 'relative support'.

What if char$(K) \neq 0$?

If $\mathcal{H}$ satisfies (Fg) support varieties via $HH^*(\mathcal{H}) = \mathrm{Ext}^*_{\mathcal{H}^e}(\mathcal{H},\mathcal{H})$.

[Linckelmann] For char$(K) = 0$, (Fg) holds for $\mathcal{H}$. ($\Rightarrow$ any M has finite complexity ($= \dim V(M)$)).

What if char$(K) \neq 0$?
Eg $\mathcal{H}_q(6)$ with char$(K) = 2$ or 3, do modules have finite complexity?

(II) A quasi-hereditary, w.r. to $(\Lambda, \leq)$.

$L(\lambda)$ simple, with projective cover $P(\lambda)$,
$\Delta(\lambda)$, $\nabla(\lambda)$ standard and costandard modules
$\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ modules with Δ, ∇ filtration.

$T(\lambda)$ (unique) indecomposable in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ with 'highest weight' λ.
$T := \oplus_\lambda T(\lambda)$ 'characteristic tilting module'.

$R_A := \mathrm{End}_A(T)^{op}$ Ringel dual of A, is QH w.r.to $(\Lambda, \leq^{op})$.

Example: $A = S_q(n,d)$ Schur algebra. View as $\text{End}_{\mathcal{H}}(V^{\otimes d})$. $V^{\otimes d} \in \text{add}(T)$ and $T(\lambda)$ occurs iff λ is ℓ-regular.

$R_A \sim_{\text{Morita}} A$ if $n \geq d$ [Donkin].
Sometimes when $n = 2$ [E-Henke, Law]

What if $3 \leq n < d$?

QH algebras have finite global dimension.

Known for $A = S_q(n,d)$ and $n \geq d$ (Totaro, Donkin), for $n = 2, 3$ (A. Parker)

What if $3 < d < n$? [A. Parker in most cases known.]

$A =$ QH (quasihereditary), T and R_A as above.

Take ${}_AP$ projective, $B := \text{End}_A(P)^{op}$ and
$F = \text{Hom}(P, -) : A - \text{mod} \to B - \text{mod}$ Schur functor.

[Rouquier] (A, P) is a QH cover of B if $F_{A-\text{proj}}$ is full and faithful ($\Leftrightarrow$ F induces iso $A \to \text{End}_B(FA)$).

EX If $n \geq d$, then $A = S_q(n, d)$ is a qh cover of $B := \mathcal{H}$. Take $P = V^{\otimes d} \cong Ae$. Then $F \cong e(-)$
takes $\nabla(\lambda) \to S^\lambda$, $\Delta(\lambda) \to (S^{\lambda'})^\#$, $P(\lambda) \to Y^\lambda$, $T(\lambda) \to (Y^{\lambda'})^\#$.

Qu Does $\mathcal{F}(\text{Sp})$ have properties similar to $\mathcal{F}(\Delta)$

THM [Hemmer-Nakano] For $\ell \geq 4$, filtration multiplicity for modules in $\mathcal{F}(\text{Sp})$ is well-defined.

(A, P) qh cover, would like F to induce iso's

$$\mathrm{Ext}^j_A(X, Y) \to \mathrm{Ext}^j_B(FX, FY)$$

(for $X, Y \in \mathcal{F}(\Delta)$).
If so for $0 \le j \le i$ then call (A, P) an $i - \mathcal{F}(\Delta)$ cover of B.

DEF [Fang-Koenig]
The Hemmer-Nakano dimension of $\mathcal{F}(\Delta)$ is the largest such i.
Then call (A, P) an $i - \mathcal{F}(\Delta)$ cover of B.

DEF $M =$ some A-module, domdim$(M) \ge n$ if M has injective resolution

$$0 \to M \to I_0 \to \ldots \to I_n \to \ldots$$

with $I_0, I_1, \ldots, I_n$ projective.

THM [Fang-Koenig]
A qh with simple preserving duality $(-)^0$
and domdim$(A) \geq 2$. Then
HN-dim $\mathcal{F}(\Delta) =$ domdim$(T) - 2$ and
domdim$A = 2$domdim$T$.

$A = S_q(n,d)$ and $n \geq d$ (assume not ss). Then

[FK/ FMiyachi] domdim$(A) = 2(p-1)(2(\ell - 1))$.

$\Rightarrow$ HN-dim $F(\Delta) = p - 3$ $(\ell - 3)$.

RK: If HN-dim $\mathcal{F}(\Delta) \geq 1$ then filtration multiplicity in $\mathcal{F}(F\Delta)$ is well-defined.

$\Rightarrow$ $\mathcal{F}$(Sp*) if $\ell \geq 4$.

Qh cover via R_A

Let $Q \in \text{add}(T)$ and $B := \text{End}_A(Q)^{op}$.

DEF $Q-$ domdim M := sup n | there are $Q_i \in \text{add}(Q)$ and an exact sequence

$$0 \to M \to Q_1 \to \ldots \to Q_n$$

which remains exact under $\text{Hom}(-, Q)$.

$Q - \text{codomdim}(M) = DQ - \text{domdim}_{A^{op}} DM$.

THM [Cruz] Let $P = \text{Hom}_A(T, Q)$.
If $Q-\text{codomdim}_A(T) \geq n \geq 2$ then (R_A, P) is an $(n-2)-\mathcal{F}(\Delta_{R_A})$-qh cover of B.

Application: $A = S_q(n, d)$ any n, d and $Q = V^{\otimes d}$. Then $B \cong \mathcal{H}/I_n$ and (R_A, P) is qh cover of B.

? what is HNdim?

THM [Cruz, E] A with simple preserving duality. Then

$$Q - \text{domdim} A = 2 \cdot Q - \text{domdim} T = 2 \cdot Q - \text{codomdim} T$$

THM [Cruz-E] $A = S_q(2, d)$ and $Q = V^{\otimes d}$. Then

$$Q - \text{domdim}(T) = \begin{cases} d/2 - 2 & \text{d even, } p = 2 \text{ (or } \ell = 2) \\ \infty & \text{ow} \end{cases}$$

COR $TL_{K,d}(\delta)$ is qh unless d is even and $p = 2$ or $\ell = 2$.

RK Let $p = 2$ or $\ell = 2$. If HN dim ≥ 1, still filtration multiplicity in $\mathcal{F}(F\Delta)$ well defined.

($d/2 - 2 \geq 1$ for $d \geq 6$. Leaves only $d = 2, 4$ (then R_A has finite type, very small)