Rexes for Double Cosets

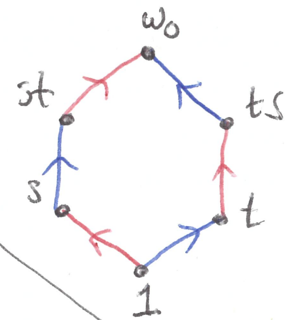
All joint w/ Hankyung Ko. Some original ideas due to Geordie, hard to extract from this presentation — long story short, he initiated, we developed!

Applications are work in progress w/ Ko, Libedinsky, Patimo.

Draw on board: How generators act on a basis.

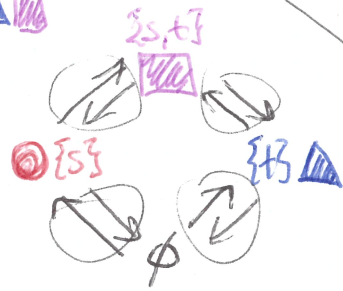
Classical: Generators

EXAMPLE: $W = S_3$

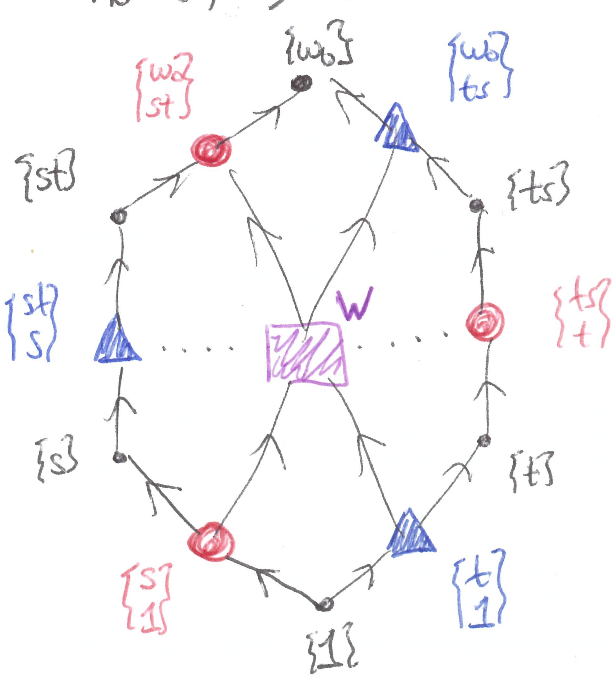


Objects: $\emptyset$

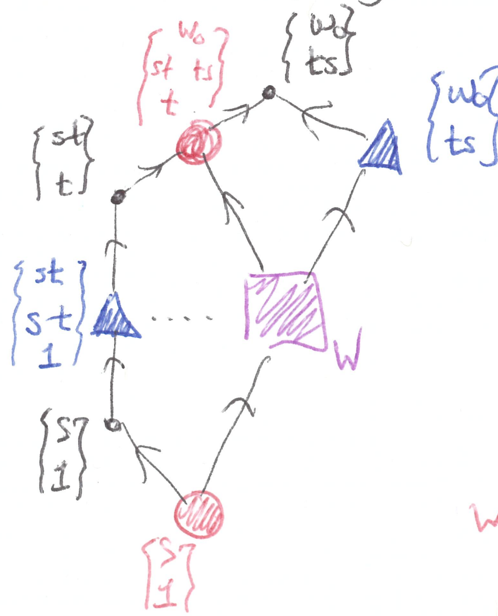
Double Coset: Generators



$\mathrm{Hom}(I, \emptyset)$ as I varies



$\mathrm{Hom}(I, \{s\})$



$\mathrm{Hom}(I, \{s,t\})$

Everything I do today works in general type, but mainly focus on S_n for simplicity.

Famously, S_n has Coxeter presentation: gens $s_i = (i \ \ i+1)$ for $i \in S = \{1, \dots, n-1\}$

$$S_n = \langle s_i \rangle \Big/ \begin{array}{ll} s_i^2 = 1 & \text{quadratic} \\ \left.\begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \quad j \neq i \pm 1 \end{array}\right\} & \text{braid} \end{array}$$

$\longleftarrow$ also a presentation for $\mathbb{C}[S_n]$

For my story it helps to discuss some variants on $\mathbb{C}[S_n]$ as well, keep braid relns but modify quadratic

$*$-monoid

$\langle s_i \rangle / \ s_i * s_i = s_i$, braid

Monoid algebra is "0-Hecke algebra"

nil Coxeter algebra

$NC(S_n) = \langle D_i \rangle / \ D_i^2 = 0$, braid

Iwahori-Hecke algebra

ALL VARIANTS FROM GEOMETRY OF FLAGS!

All these have a basis $\Longleftrightarrow S_n$, defined as follows: (focus on NC)

$\underline{w} = (i_1, i_2, \dots, i_d)$ a word in S, what I call an expression.

$\underline{w}$ is reduced expression or rex for w if $w = s_{i_1} \cdots s_{i_d}$ and no shorter word suffices, $\ell(w) = d$.

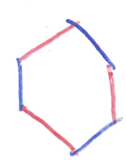
$D_{\underline{w}} := D_{i_1} D_{i_2} \cdots D_{i_d}$ for any expr.

Def: $D_w := D_{\underline{w}}$ when $\underline{w}$ a rex for w.

Well-defined (indep of $\underline{w}$) b/c:

Thm (Matsumoto): Any two rexes for w connected by braid relns.

All variants have "same behavior" for rexes. Differ in how to interpret non-rexes.

GO TO PICTURE AND DISCUSS WALKS AROUND .

NC has nice feature of isolating rexes as special:

$$D_{\underline{w}} \neq 0 \iff \underline{w} \text{ is a rex.}$$

We'll generalize (each variant) to a category w/ objects $I \subset S$ ③

to which we associate $W_I = \langle s_i \mid i \in I \rangle$

Ex: In S_n, $W_I \cong S_{n_1} \times \dots \times S_{n_k}$ where $\sum n_k = n$.

In general we only consider I with W_I finite $\rightsquigarrow$ has longest element w_I.

In our category $\mathrm{Hom}(I,J)$ has basis $\leftrightarrow$ ${}_{W_J}\backslash W / W_I$ (should reverse because composition R to L annoying and I may forget for pedagogy...)

$\mathrm{Hom}(\emptyset, \emptyset)$ = previous algebra

composition $(\emptyset \leftarrow s \leftarrow \emptyset) \longleftrightarrow$ generator

Generators: $I \rightleftarrows I \sqcup \{s\} =: Is$

shorten $[\emptyset, s, \emptyset]$

this composition is never invertible. Think: $[s, \emptyset]$ is not a permutation but a map from $\{1, \dots, n\}$ to $\{\{1,2\}, 3, \dots, n\}$

Can do for $\mathbb{C}[S_n]$ but need to change generators, will instead focus on NC and $*$.

Need to
1) Motivate
2) Describe
3) State awesome theorems.

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2) Is easy in $*$-monoid variant, SC. We'll say how generator acts on basis of double cosets. If $p \in {}_{W_I}\backslash W / W_J$ then $\exists!\ q \in {}_{W_{Is}}\backslash W / W_J$

with $p \subset q$. $[Is, I] * p = q$.

Conversely, q splits into ≥ 1 coset for ${}_{W_I}\backslash W / W_J$, and one is maximal in bruhat order. E.g. $\overline{q} \in q$ is ! max'l element, then $p \subset q$ maximal if $\overline{p} = \overline{q}$.

$[I, Is] * q = p$.

SHOW OFF IN PICTURE.

Aside: Easiest way to describe mult. of elements is $SC \cong \mathrm{Kar}(*\text{-monoid})$

Idempotents $= \{w_I\}_{I \subset S}$

~~Def~~ **Def:** A <u>double coset expression</u> $\underline{I} = [I_0, I_1, \ldots, I_d]$ is

a composition of generators, i.e. $I_{i+1} = I_i s$ or $I_i = I_{i+1} s$ for some s.

Can assign a <u>length</u>, $\ell([I, Is]) = \ell([Is, I]) = \ell(Is) - \ell(I)$ where

$\ell(J) = \ell_W(w_J)$. DISCUSS LENGTHS ON PIC.

Can assign a length to double coset, $\ell(p) = 2\,\ell_W(\bar{p}) - \ell(I) - \ell(J)$

$p \in W_I \backslash W / W_J$

<u>Note!</u> For $p = \{w\} \in W_\emptyset \backslash W / W_\emptyset$, $\ell(p) = 2\ell_W(w)$.

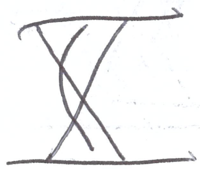
<u>Def:</u> $\underline{I}$ is <u>reduced</u> if express p with $\ell(\underline{I}) = \ell(p)$. (One of many equiv defns.)

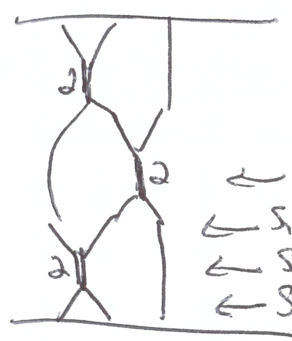
In picture, rexes are oriented paths from base.

<u>base</u> = $W_I \backslash W / W_I$ coset containing 1. $\ell(\text{base}) = 2\ell_W(w_I) - \ell(I) - \ell(I) = 0$.

.... is not reduced in either direction. (Maybe do root hyperplanes here!)

For those who like webs, in type A can draw expressions.

Old sts =  new $[\emptyset, \{s\}, \emptyset, \{t\}, \emptyset, \{s\}, \emptyset]$

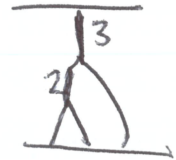


$\leftarrow S_1 \times S_2$

$\leftarrow S_1 \times S_1 \times S_1$

$\leftarrow S_2 \times S_1$

$\leftarrow S_1 \times S_1 \times S_1$

and $[\{s,t\}, \{s\}, \emptyset]$ is 

<u>Geom/Alg Source</u> Easiest to explain using NC first.

~~Note! In $*$-version, this category is Karoubi envelope of $*$-monoidal I-idempotents $\leftrightarrow$ W_I~~

Setup:

$$\begin{tikzcd}
 & Fl(0,3) = pt. & \\
Fl(0,2,3) \arrow[ur, "\mathbb{P}^2"] & & Fl(0,1,3) \arrow[ul, "\mathbb{P}^2"'] \\
 & Fl(0,1,2,3) \arrow[ul, "\mathbb{P}^1"] \arrow[ur, "\mathbb{P}^1"'] \arrow[uu, "\text{forget}"] &
\end{tikzcd}
\qquad \text{equals} \qquad
\begin{tikzcd}
 & G/G & \\
G/P_1 \arrow[ur] & & G/P_2 \arrow[ul] \\
 & G/B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\{0 \subset P \subset \mathbb{C}^3\} = Fl(0,2,3)$

$Fl(0,1,2,3) = \{0 \subset L \subset P \subset \mathbb{C}^3\}$

$G = GL_3 \qquad B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$P_1 = \begin{pmatrix} \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & \begin{matrix} * \\ * \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & * \end{pmatrix} \qquad P_2 = \begin{pmatrix} * & \begin{matrix} * & * \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} \end{pmatrix}$

To this setting one can apply all sorts of functors, e.g. $H^*(-)$.

$$\begin{tikzcd}
 & H^*(G/G) \arrow[dl, shift left] \arrow[dr, shift left] \arrow[dd, shift left] & \\
H^*(G/P_1) \arrow[ur, shift left] \arrow[dr, shift left] & & H^*(G/P_2) \arrow[ul, shift left] \arrow[dl, shift left] \\
 & H^*(G/B) \arrow[ul, shift left] \arrow[ur, shift left] \arrow[uu, shift left] &
\end{tikzcd}$$

1) Draw pullbacks, they're inclusions, ring homoms.

2) Discuss pushforward below.

3) Draw pushforwards, they're surjections, NOT ring hom.

4) Diagram commutes.

When $Y \xrightarrow[\pi]{\mathbb{P}^k} X$ have $H^*(Y) \xrightarrow{\pi_*} H^{*-2k}(X)$ "integrate along fibers"

Can do this algebraically:

$$\begin{tikzcd}
 & R^{S_3} \arrow[dl, shift left] \arrow[dr, shift left] \arrow[dd, shift left] & \\
R^s \arrow[ur, shift left, "\partial^s_{st}"] \arrow[dr, shift left] & & R^t \arrow[ul, shift left, "\partial^t_{st}"'] \arrow[dl, shift left] \\
 & R = \mathbb{C}[x_1,x_2,x_3] \arrow[ul, shift left, "\partial_s"] \arrow[ur, shift left, "\partial_t"'] \arrow[uu, shift left, "\partial_{S_3}"] &
\end{tikzcd}$$

$R^s = \mathbb{C}[x_1+x_2, x_1x_2, x_3]$

$R = \mathbb{C}[x_1,x_2,x_3] \circlearrowleft S_3$

(to get above picture kill ideal $(R^{S_3}_+)$.

(this is really for
$$\begin{tikzcd}
 & */G & \\
*/P_1 \arrow[ur] & & */P_2 \arrow[ul] \\
 & */B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$
(equiv cohom.))

The maps ∂_s are Demazure operators. $\partial_s(f) = \dfrac{f - sf}{x_1 - x_2}$, denom divides num.

Ex! $\partial(x_1^3) = \dfrac{x_1^3 - x_2^3}{x_1 - x_2} = x_1^2 + x_1x_2 + x_2^2$.

Similarly, $\partial_{S_3} = \dfrac{\sum_{w \in S_3} \operatorname{sign}(w)\, w(f)}{\prod_{i<j} (x_i - x_j)}$

Usual viewpoint on NC:

$$\begin{tikzcd} R \arrow[r, two heads, "\partial_i"] \arrow[rr, bend right, "D_i"'] & R^{s_i} \arrow[r, hook] & R \end{tikzcd}$$

<u>Thm</u> (Demazure): $DNC(S_n) \xrightarrow{\sim}$ subalg of $\operatorname{End}_{\mathbb{C}}(R)$ gen by D_i

2) $\partial_W = D_{w_0}$ (etc for other parabolic subgps.)

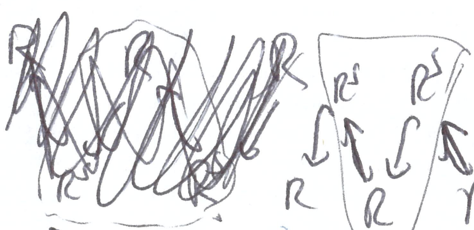
$\boxed{\text{Rmk: Do } K \text{ instead, get } *\text{-monad!}}$

<u>Aside</u>: D_w very important in Schubert calculus.

So: what if we study all maps ∂_I^J and see what part of $\operatorname{Hom}_{\mathbb{C}}(R^K, R^L)$ they generate. This is $\mathcal{NC}(S_n)$, non-full subcat of Vect with objects R^I, $I \subset S$.

Observe: All relations in $\mathcal{NC}(S_n)$ come from more <u>local</u> relations.

<u>Ex 1</u>: $D_s^2 = 0$



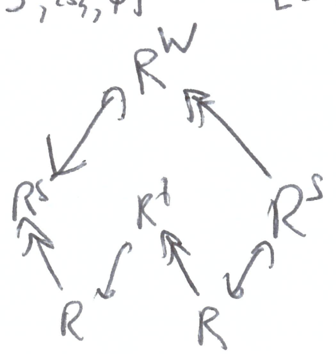
but D_s kills f if $f \in R^s$, i.e. $D_{[s,\emptyset,s]} = 0$ already.

$D_{[\emptyset, s, \emptyset, s, \emptyset]}$

<u>Ex 2</u>: $D_{sts} = D_{tst}$ DEMONSTRATE ON BOARD. 4 more local relations!

Clearly $D_{[S, \{s\}, \emptyset]} = D_{[S, \{t\}, \emptyset]}$ pushforward $\circ$ pushforward $=$ pushforward

Interesting!



$D_{[s,S,s]}$ high road

$\|$

$D_{[s,\emptyset,t,\emptyset,s]}$ low road

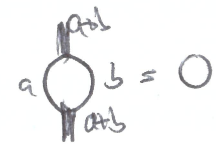
Switchback relation

Thm (EK) (se)

1) $\mathcal{NC}(W)$ has presentation w/ relns

a) "quadratic" $[I_s, I, I_s] = 0$

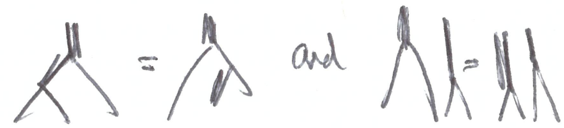
for se ($[I_s, I, I_s] = [I_s]$)



braid

b) $[I_{st}, I_s, I] = [I_{st}, I_t, I]$

and reverse

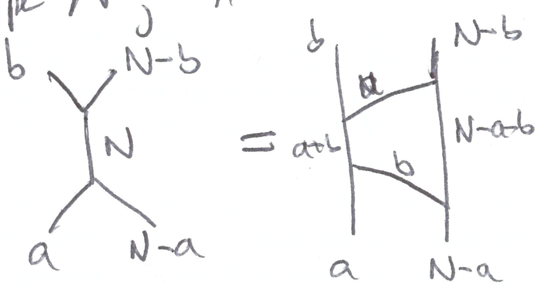


c) switchback. One such relation for each irred finite Cox gp and each pair $s,t \in S$ with $t \neq w_0 s w_0$

general idea

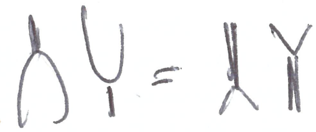
Both rexes for $W_{\hat{i}} \backslash S_n / W_{\hat{j}}$ coset containing w_0

In type A, if $a+b<N$ then



and horiz flip.

Together with distant switchbacks like



2) $\partial_{\underline{I}} = 0 \iff \underline{I}$ is not reduced

3) "Matsumoto": All rexes for p related by braid relations (so ∂_p defined)

4) $\{\partial_p\}_{p \in W_I \backslash W / W_J}$ a basis for $\mathrm{Hom}_{re}(R^J, R^I)$

5) $\coprod_{n \geq 0} \mathcal{NC}(S_n) \cong \mathrm{AssGr}(\text{Type A Webs})$ for $\mathfrak{gl}_\infty$

6) Bruhat order via "subexpressions"

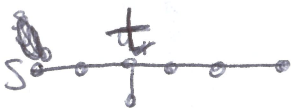
Remarks: 1) Defn of rex equivalent to Williamson's but much less technical.

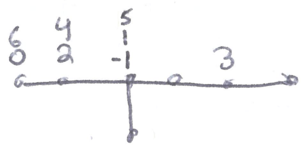
2) $[I_0 \subset \dots \subset K_1 \supset \dots \supset I_1 \subset \dots \subset K_2 \supset \dots \supset I_2 \subset \dots]$ is reduced iff lengths add in $w_{K_1}(w_{I_1}^{-1} w_{K_2})(w_{I_2}^{-1} w_{K_3}) \dots$

3) Higher braid relns (e.g. 3-cells, etc) not known.

4) In any type get switchback via switchback leapfrog

$u_{-1} = w_0 t w_0 \quad u_0 = s \quad u_1 = w_{\hat{s}} u_{-1} w_{\hat{s}} \quad u_2 = w_{\hat{u}_1} s w_{\hat{u}_1} \quad u_3 = w_{\hat{u}_2} u_1 w_{\hat{u}_2} \quad \dots$

Ex: 

then 

6-periodic

$$[\hat{s}, S, \hat{t}] = [\hat{u}_0, \hat{u}_0\hat{u}_1, \hat{u}_1, \hat{u}_1\hat{u}_2, \hat{u}_2, \hat{u}_2\hat{u}_3, \hat{u}_3, \hat{u}_3\hat{u}_4, \hat{u}_4, \hat{u}_4\hat{u}_5, \hat{u}_5]$$

How far do you go? It depends - could be longer or shorter than period! But ends in $\hat{t}$.

Applications: 1) Braid relns fundamental to Hecke category + light leaves basis.

KGLP: light leaves for singular Soergel bimodules.

2) Specifically, $\partial_{[\hat{s}, S, \hat{t}]} = \partial_{\underline{I}} \iff R^{\hat{s}} \otimes_{R^S} R^{\hat{t}} \overset{\oplus}{\subset} BS(\underline{I})$

3) Presentation of Hecke algebroid should come out of this easily.

4) Rex combinatorics of a completely new type. high roads, low roads, atomic cosets...

5) Likely related to Stanley's funny rex counting results.

2') (Williamson's original point) $BS(\underline{I})_{Var}$ as resolution of sing. of $\overline{O_p}$.

Things that didn't make it into this talk

1) If $t = w_0 s w_0$ then $[\hat{s}, S, \hat{t}]$ is unique rex. for $\bar{p} = w_0$

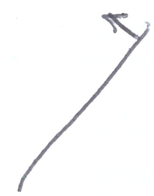
2) Redundancy subgps. f, Howlett.

3) Using PD to define pushforward.

4) $P_I \backslash G / P_J \longleftrightarrow W_I \backslash W / W_J$ Coset bruhat.

5) Subexpressions as strolls

6) Reduced via idea of ~~oo~~ root hyperplanes



Never do