Full runner removal theorem for Ariki-Koike algebras

OIST Workshop:
Representation Theory of Hecke Algebras and Categorification

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9 June 2023

Iwahori-Hecke and Ariki-Koike algebras

	H_n	$\mathcal{H}_{r,n}$
	simple modules	**simple modules**
ordinary	$\{S^\lambda,\ \lambda$ partition of $n\}$	$\{S^{\boldsymbol\lambda},\ \boldsymbol\lambda$ r-multipartition of $n\}$
modular	$\{D^\lambda,\ \lambda \vdash n$ e-regular$\}$	$\{D^{\boldsymbol\lambda},\ \boldsymbol\lambda \vdash n$ Kleshchev$\}$

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QUESTION	$[S^\lambda : D^\mu] =$ **?**	$[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}] =$ **?**

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char 0	**LLT algorithm**	**LLT-type algorithm**

LLT algorithm for H_n

LLT algorithm computes the decomposition numbers for H_n

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- computation of the **canonical basis elements** of $\mathcal{F}$:

$$G(\mu) = \mu + \sum_{\lambda \lhd \mu} d^e_{\lambda\mu}(q)\lambda, \qquad d^e_{\lambda\mu}(q) \in q\mathbb{N}[q]$$

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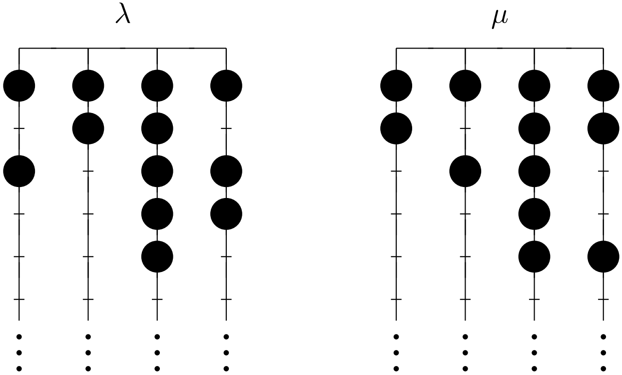
- **Theorem** (Ariki). $[S^\lambda : D^\mu] = d^e_{\lambda\mu}(1)$.

Full runner removal - example $r = 1$

Suppose $e = 4$. Consider the following abacus displays:



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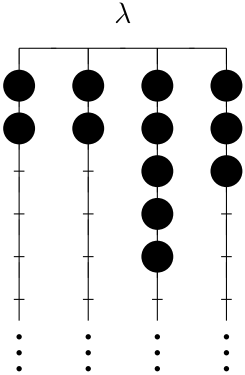

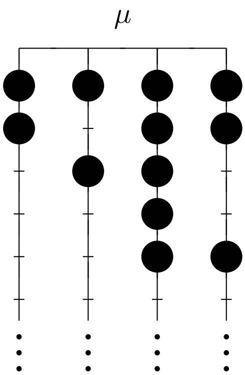

Full runner removal - example $r = 1$

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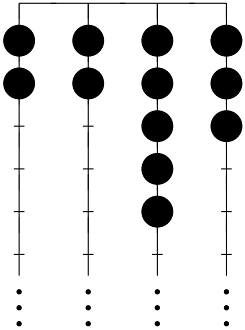
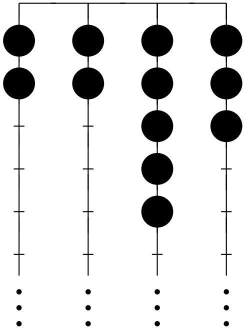
λ



μ



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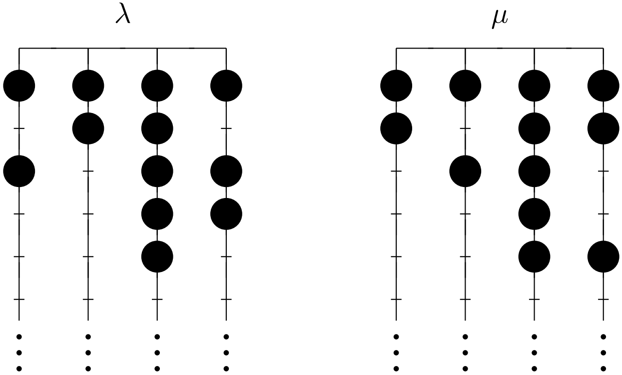

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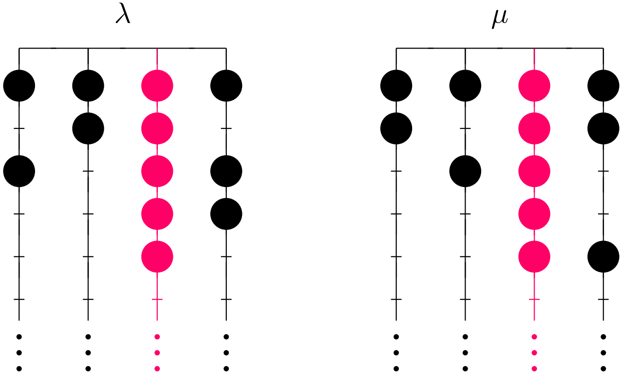
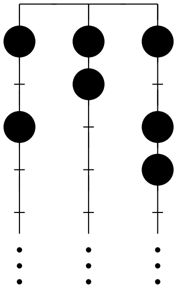
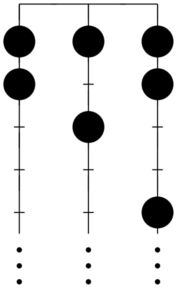


Full runner removal - example $r = 1$

λ | $e = 4$ | μ
$\downarrow$ | runner removal | $\downarrow$
λ^- | $e = 3$ | μ^-



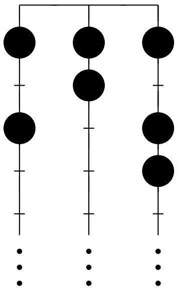


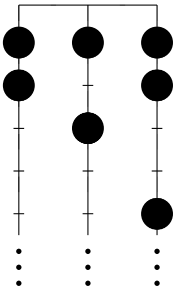
Full runner removal - example $r = 1$

λ $\downarrow$ λ^-

$e = 4$
runner removal
$e = 3$

μ $\downarrow$ μ^-





$$d^4_{\lambda\mu}(q) = d^3_{\lambda^-\mu^-}(q)$$

Full Runner Removal - level 1

Theorem (Fayers)

*Let $e \geq 3$. Let λ, $\mu \vdash n$ be in the same block with a "**long enough**" runner i. Let λ^-, μ^- as before.*

Then

$$d^e_{\lambda\mu}(q) = d^{e-1}_{\lambda^-\mu^-}(q).$$

Full runner removal theorem - level $r > 1$

Theorem (D.)

Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be r-multipartitions of n in the same block with $\boldsymbol{\mu}$ e-multiregular.

Let $\boldsymbol{s} \in (\mathbb{Z}/e\mathbb{Z})^r$ be a multicharge. Let $\boldsymbol{k} \in \mathbb{Z}^r_{\geq 0}$.
Denote by $^{+\boldsymbol{k}}$ the addition of a runner full of beads in each component.

*If the new inserted runners are "***long enough***", then*

$$d^{e+1}_{\boldsymbol{\lambda}^{+\boldsymbol{k}}\boldsymbol{\mu}^{+\boldsymbol{k}}}(q) = d^{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q).$$

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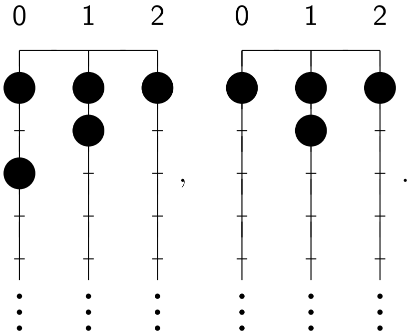
Ingredients:

- Addition of a full runner for $\mathcal{H}_{r,n}$
- LLT-type algorithm for $\mathcal{H}_{r,n}$

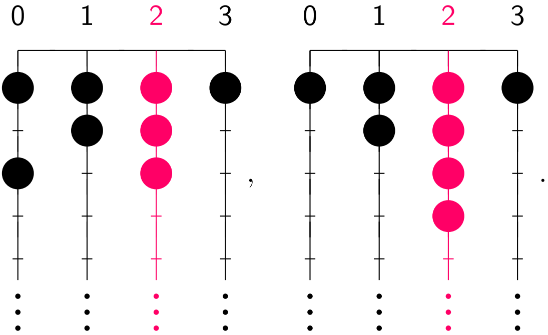
Addition of a full runner - level 2

$$\boldsymbol{\mu} = ((2,1),(1)), \quad e = 3$$



Addition of a full runner - level 2

$$\boldsymbol{\mu}^{+(6,10)} = ((3,2,1^2),(7,4,1^2)), \quad e = 4$$



Sketch of the proof

$r = 2$, $\boldsymbol{\mu} = (\mu^{(1)}, \mu)$ e-multiregular, $k^{(1)}, k^{(2)} \in \mathbb{Z}_{\geq 0}$ such that

$$k^{(2)} - k^{(1)} \geq \mu_1 + e - 1.$$

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$$\implies G_{e+1}(\mu^{+k^{(2)}}) = \mu^{+k^{(2)}} + \sum_{\mu \rhd \lambda} d^e_{\lambda\mu}(q)\lambda^{+k^{(2)}}$$

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$$G^{\boldsymbol{s}^+}_{e+1}((\varnothing, \mu^{+k^{(2)}})) = (\varnothing, \mu^{+k^{(2)}}) + \sum_{\mu \rhd \lambda} d^e_{\lambda\mu}(q)(\varnothing, \lambda^{+k^{(2)}})$$

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Sketch of the proof

STEP 4.

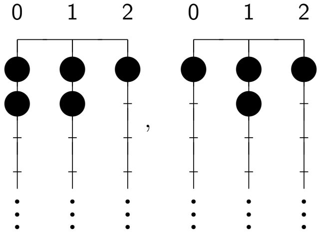
e | $e+1$

LLT alg: $G^{(s_1)}(\mu^{(1)}) = \mathfrak{f} \cdot \varnothing$

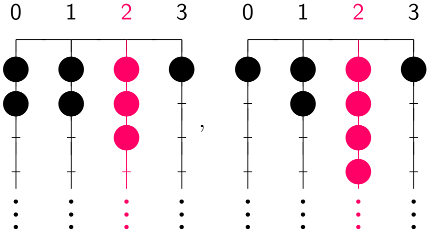
$\mathfrak{f} G_e^{\boldsymbol{s}}((\varnothing, \mu))$ $\leftrightsquigarrow$ Translation from $\mathfrak{f}$ to $\mathfrak{F}$

$\mathfrak{F} G_{e+1}^{\boldsymbol{s}^+}((\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}}))$

$f_0 f_1 f_2(\varnothing, \mu)$



$\mathfrak{F}_0 \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3(\varnothing^{+k^{(1)}}, \mu^{+k^{(2)}})$



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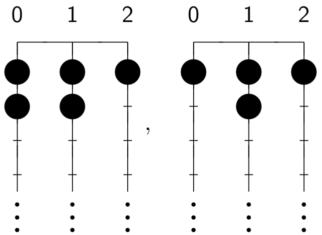

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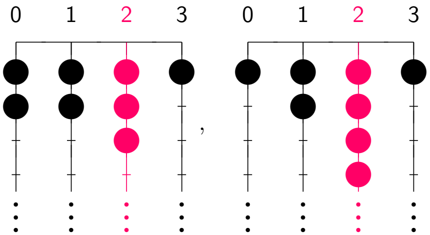

$$G_e^{\boldsymbol{s}}(\boldsymbol{\mu})^{+\boldsymbol{k}} = G_{e+1}^{\boldsymbol{s}^+}(\boldsymbol{\mu}^{+\boldsymbol{k}}).$$

Thank you for your time!