

Decomposable Specht modules

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Abstract

I will give a brief survey of the study of decomposable Specht modules for the symmetric group and its Hecke algebra, which includes results of Murphy, Dodge and Fayers, and myself. I will then report on an ongoing project with Louise Sutton, in which we are studying decomposable Specht modules for the Hecke algebra of type B indexed by ‘bihooks’.

1 Decomposable Specht modules in level 1

Let \mathbb{F} be a field of characteristic $p \geq 0$ throughout.

The Specht modules $\{S^\lambda \mid \lambda \vdash n\}$ over \mathfrak{S}_n are the ordinary irreducible \mathfrak{S}_n -modules, indexed by partitions λ of n .

We have the following fundamental fact about Specht modules.

Theorem 1.1 [9, Corollary 13.18]. *If $p \neq 2$ or λ is 2-regular, then S^λ is indecomposable.*

When $p = 2$ and λ is 2-singular, it is a difficult problem to determine whether or not S^λ is decomposable. However, some special cases are very tractable.

Theorem 1.2 [11, Theorems 4.1 and 4.5]. *Let $\lambda = (a, 1^b)$, with $a + b = n$. If n is even, then S^λ is indecomposable.*

If n is odd and $n \geq 2b$, then S^λ is indecomposable if and only if $a - b - 1 \equiv 0 \pmod{2^L}$, where $2^{L-1} \leq b < 2^L$.

Given that S^λ is decomposable if and only if $S^{\lambda'}$ is, where λ' is the conjugate of λ , the restriction that $a \geq b$ is in fact not a problem, and Murphy’s result gives a complete classification of which Specht modules indexed by hook partitions are decomposable.

Note that as n gets larger, so does the proportion of Specht modules indexed by hook partitions which are decomposable.

Example. Let $n = 13$. Then we have the following (and their conjugates).

λ	S^λ is
(13)	indecomposable
(12, 1)	indecomposable
(11, 1 ²)	indecomposable
(10, 1 ³)	decomposable
(9, 1 ⁴)	decomposable
(8, 1 ⁵)	decomposable
(7, 1 ⁶)	indecomposable

Similarly, when $n = 21$, only 5 of the 17 interesting Specht modules are indecomposable.

30 years later, Dodge and Fayers found the first new examples of decomposable Specht modules since Murphy:

Theorem 1.3 [7, Theorem 3.1]. *Suppose $\lambda = (a, 3, 1^b)$ with $a \geq 4$ and $b \geq 2$. Then S^λ is decomposable if at least one of the following holds:*

- $a + b \equiv 0$ or $2 \pmod{8}$, $a \geq 6$ and $b \geq 4$;
- $a + b \equiv 2 \pmod{4}$ and $\binom{a+b-3}{a-3}$ is odd;
- $a + b \equiv 0 \pmod{4}$ and $\binom{a+b-9}{a-5}$ is odd.

A natural generalisation of this problem is to instead consider Specht modules over the Iwahori–Hecke algebra of the symmetric group. This is the unital, associative \mathbb{F} -algebra \mathcal{H}_n with generators T_1, T_2, \dots, T_{n-1} and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for all } i, \\ T_i T_j &= T_j T_i && \text{for } |i - j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 0 \leq i \leq n - 2, \end{aligned}$$

where $q \in \mathbb{F}$ is a primitive e th root of unity.

Now the Specht modules $\{S^\lambda \mid \lambda \vdash n\}$ over \mathcal{H}_n are the ordinary irreducible \mathcal{H}_n -modules, indexed by partitions λ of n .

As for symmetric groups, we have (following [5, Theorem 3.5]) that S^λ is decomposable if and only if $S^{\lambda'}$ is and:

Theorem 1.4 [6, Corollary 8.7]. *If $e \neq 2$ or λ is 2-regular, then S^λ is indecomposable.*

Once again, when $e = 2$ (i.e. $q = -1$), and λ is 2-singular, it is difficult to determine whether or not S^λ is decomposable.

Shortly after Dodge and Fayers obtained their results, we extended Murphy's result to \mathcal{H}_n .

Theorem 1.5 [13, Theorem 6.12]. *Suppose $p \neq 2$ and $\lambda = (a, 1^b)$. Then S^λ is indecomposable if and only if n is even or $b = 2$ or 3 with $p \mid \lceil \frac{a}{2} \rceil$.*

2 KLR algebras

We may further generalise our setting to cyclotomic Hecke algebras, deformations of the complex reflection groups $G(l, 1, n) = \mathbb{Z}/l\mathbb{Z} \wr \mathfrak{S}_n$. For our purposes, the following theorem of Brundan and Kleshchev will provide the perspective we take in looking for decomposable Specht modules.

Theorem 2.1 [3, Main Theorem]. *The (integral) cyclotomic Hecke algebra in quantum characteristic $e \geq 2$ is isomorphic to a level l cyclotomic Khovanov–Lauda–Rouquier algebra \mathcal{R}_n^Λ of type $A_{e-1}^{(1)}$ if $e < \infty$, or A_∞ if $e = \infty$ (i.e. corresponding to dominant weight $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2} + \cdots + \Lambda_{\kappa_l}$).*

The cyclotomic KLR algebra \mathcal{R}_n^Λ is a unital, associative \mathbb{F} -algebra with generators

$$\{e(i) \mid i \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, y_2, \dots, y_n\} \cup \{\psi_1, \psi_2, \dots, \psi_{n-1}\}$$

subject to a long list of relations. This algebra is naturally \mathbb{Z} -graded, which leads us to studying the graded representation theory of cyclotomic Hecke algebras.

2.1 Specht modules over \mathcal{R}_n^Λ

There is a theory of Specht modules over cyclotomic Hecke algebras which naturally lead to Specht modules over \mathcal{R}_n^Λ , which are the ordinary irreducibles. In fact, we recently constructed an analogous Specht module theory for \mathcal{R}_n^Λ in type C in [2]. We are mostly interested in the non-semisimple story.

Theorem 2.2. *Let $\alpha_{i,n}^\vee = \alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_{i+n-1}^\vee$.*

[1] In type $A_{e-1}^{(1)}$ or A_∞ , \mathcal{R}_n^Λ is semisimple if and only if $e > n$ and $\langle \Lambda, \alpha_{i,n}^\vee \rangle \leq 1$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.

[14, Theorem 1.1] In type $C_{e-1}^{(1)}$ or C_∞ , \mathcal{R}_n^Λ is semisimple if and only if $\langle \Lambda, \alpha_{i,n}^\vee \rangle \leq 1$ for all $i \in \mathbb{Z}/e\mathbb{Z}$ and $\frac{n-1}{2} \leq \bar{\kappa}_j \leq e - \frac{n+1}{2}$ for all $1 \leq j \leq l$.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)})$ be an l -multipartition of n and let T^λ denote the *column initial* λ -tableau, and denote by i^λ its residue sequence modulo e .

Example. Let $\lambda = ((4, 3), (3, 2, 1))$. Then

$$T^\lambda = \begin{array}{cccc} \boxed{7} & \boxed{9} & \boxed{11} & \boxed{13} \\ \boxed{8} & \boxed{10} & \boxed{12} & \\ \\ \boxed{1} & \boxed{4} & \boxed{6} & \\ \boxed{2} & \boxed{5} & & \\ \boxed{3} & & & \end{array}$$

For a λ -tableau T , define the permutation w^T to be the permutation satisfying $w^T T^\lambda = T$.

Following [10], the Specht module S^λ is the cyclic \mathcal{R}_n^Λ -module with homogeneous generator z^λ subject to the following relations.

1. $e(i)z^\lambda = \delta_{i,i^\lambda} z^\lambda$;
2. $y_r z^\lambda = 0$ for all r ;

3. $\psi_r z^\lambda = 0$ whenever r and $r + 1$ are in the same column of \mathbb{T}^λ ;
4. Garnir relations.

For each $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{i_1} \dots s_{i_r}$, and define the corresponding element $\psi_w := \psi_{i_1} \dots \psi_{i_r} \in \mathcal{R}_n^\Lambda$. In general these elements depend on the choice of reduced expression, since the ψ generators do not satisfy braid relations! Finally, let $\psi^\mathbb{T} = \psi_{w^\mathbb{T}}$.

Theorem 2.3 ([4, 10]). *Let λ be an l -multipartition of n . The Specht module S^λ is graded, with homogeneous basis*

$$\{v^\mathbb{T} := \psi^\mathbb{T} z^\lambda \mid \mathbb{T} \in \text{Std}(\lambda)\}.$$

Theorem 2.4 ([12, 8]). *If $e \neq 2$ and $\kappa_i \neq \kappa_j$ for all $i \neq j$, or if λ is a conjugate Kleshchev multipartition, then S^λ is indecomposable.*

It is natural to now look for decomposable Specht modules in higher levels. Our presentation and basis allow us to calculate endomorphisms of Specht modules, as any $\varphi \in \text{End}(S^\lambda)$ satisfies

$$\varphi(z^\lambda) = \sum a_\mathbb{T} v^\mathbb{T} \text{ for some } a_\mathbb{T} \in \mathbb{F},$$

where we sum over all $\mathbb{T} \in \text{Std}(\lambda)$ such that $\text{res } \mathbb{T} = i^\lambda$, and the right-hand side must satisfy the defining relations of S^λ .

3 Decomposable Specht modules in level 2

We now fix $l = 2$, so that $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2}$ and \mathcal{R}_n^Λ is isomorphic to a Hecke algebra of type B , and assume that $p \neq 2$.

For now, we fix $e \geq 3$ and $\Lambda = 2\Lambda_0$. We study Specht modules indexed by *bihooks* $\lambda = ((a, 1^b), (c, 1^d))$, a natural generalisation of hooks in level 1.

Theorem 3.1 ([15]). *[Small bihooks] Let $n \leq 2e$. Then S^λ is decomposable if and only if $n = 2e$ and $\lambda = ((a, 1^b), (a, 1^b))$ for some a, b .*

Proof. It is easy to check that if $\lambda = ((a, 1^b), (a, 1^b))$ is a bipartition of $n < 2e$, λ is conjugate Kleshchev, and is thus indecomposable. In all other cases, we deduce indecomposability by looking at the few tableaux of the correct residue and showing that there cannot be a non-trivial endomorphism. If $n = 2e$ and $\lambda = ((a, 1^b), (a, 1^b))$ for some a, b , there is an endomorphism given by swapping the two components of \mathbb{T}^λ . A long calculation shows that this endomorphism is an idempotent. \square

The following is still work in progress, but will become a theorem soon!

Conjecture 3.2 ([15]). *Let $\lambda = ((k), (k))$. Then S^λ is decomposable if and only if $k \geq e$.*

Using some tricks with i -induction and i -restriction, we show the following.

Theorem 3.3 ([15]). *Let $k \geq 1$, $0 \leq a \leq e - 1$ and $0 \leq b \leq e - 1$ with $a + b \neq e$. The Specht module $S^{((ke), (ke))}$ is decomposable if and only if $S^{((ke+a, 1^b), (ke+a, 1^b))}$ is.*

Thus proving Conjecture 3.2 will yield a large family of decomposable Specht modules.

Example. Let $e = 3$ and $\lambda = ((6), (6))$. There are six standard λ -tableaux with residue sequence i^λ , obtained by permuting the e -bricks.

$$\begin{array}{lll}
 \mathbf{T}^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array} & \mathbf{R} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 10 & 11 & 12 \\ \hline 1 & 2 & 3 & 7 & 8 & 9 \\ \hline \end{array} & \mathbf{S} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline \end{array} \\
 \\
 \mathbf{T} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \end{array} & \mathbf{U} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 8 & 9 \\ \hline 4 & 5 & 6 & 10 & 11 & 12 \\ \hline \end{array} & \mathbf{W} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & 10 & 11 & 12 \\ \hline \end{array}.
 \end{array}$$

There is a homomorphism given by $\varphi(z^\lambda) = 4v^{\mathbf{R}} + 2v^{\mathbf{S}} + 2v^{\mathbf{T}} + v^{\mathbf{U}}$.

It can be shown that $v^{\mathbf{S}} - v^{\mathbf{T}}$ and $v^{\mathbf{W}}$ are eigenvectors for this endomorphism, with eigenvalues -4 and -6 . Thus there are at least two distinct generalised eigenspaces, and S^λ is decomposable.

Future work: find families of decomposable Specht modules when $e = 2$ and $\Lambda = 2\Lambda_0$ or $\Lambda_0 + \Lambda_1$.

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