Superalgebra Deformations of Webs

Lightning Talk!

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Rep Theory of Hecke Algebras & Categorification Workshop

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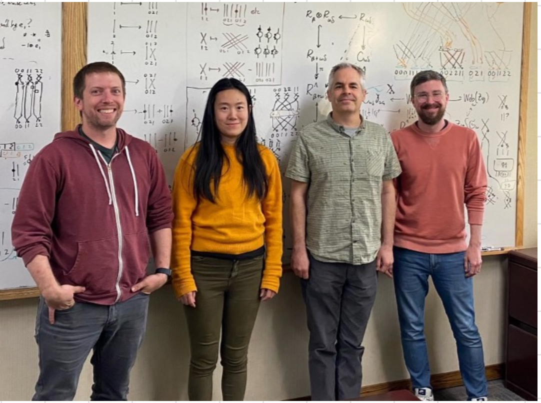
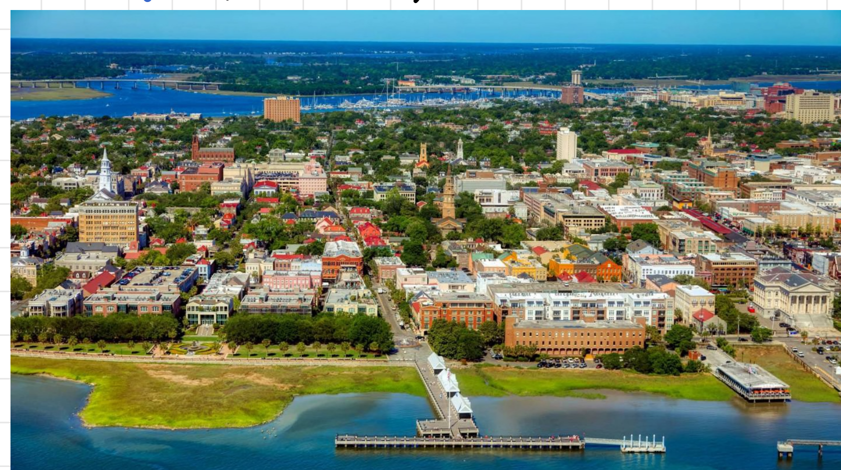


jt. with Jon Kujawa, Rob Muth, Jieru Zhu

Charleston



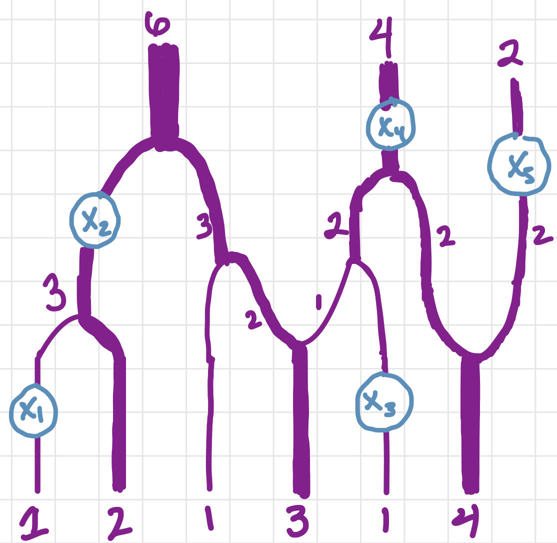
Superalgebra Deformed

Web Diagrams

$\Bbbk$ = char 0 domain

$A = A_{\bar 0} \oplus A_{\bar 1}$ $\Bbbk$-superalgebra

Step 1: Take some parallel strings of varying thickness ($\in \mathbb{N}$), and (working bottom to top), merge and split them



$: (1,2,1,3,1,4) \longrightarrow (6,4,2)$

$x_1, x_3 \in A$

$x_2, x_4, x_5 \in A_{\bar 0}$.

Step 2 Attach coupons from A to the strands, so that thick strands only have coupons from $A_{\bar 0}$.

Webs as a Category

morphisms are $\mathbb{Z}_2$-graded $\Bbbk$-modules

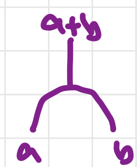
Definition. Web_A is the **monoidal supercategory** with

- **Objects**: • Sequences of non-negative integers $\vec{a} = (a_1, \ldots, a_k)$

- **Morphisms**
  - A morphism $\vec{a} \to \vec{b}$ is a $\Bbbk$-linear combo of Web-diagrams
  - Generated by

$: a+b \to (a,b)$

parity $= \bar{0}$



$: (a,b) \to a+b$

parity $= \bar{0}$

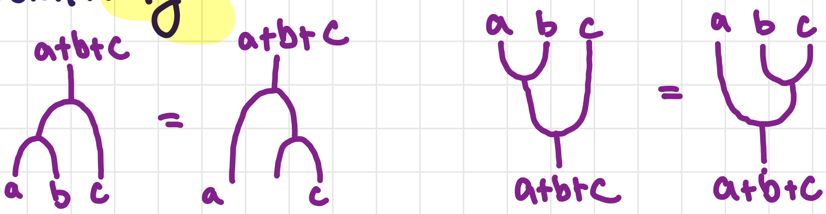
$: a \to a$

$x \in A$ or $A_{\bar{0}}$

parity $= |x|$

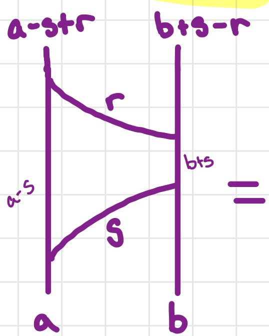
Web Relations

• Associativity

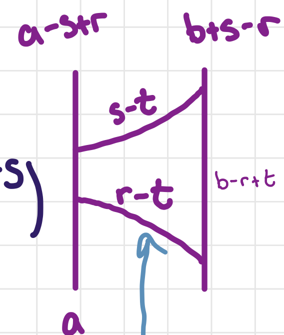


• "Rung-Swap"

For $a, b, r, s \geq 0$:

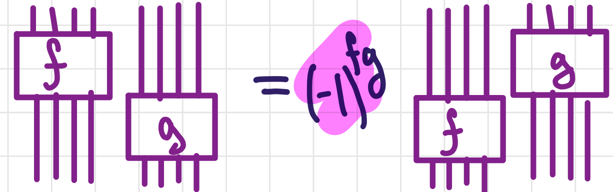


$$= \sum_{t=0}^{\infty} \binom{a-b+r-s}{t}$$

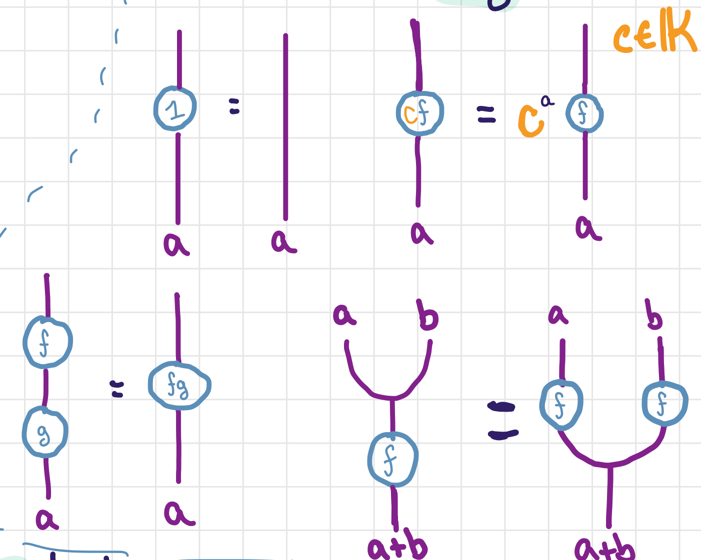


Diagrams with Negative Thickness strands are 0.

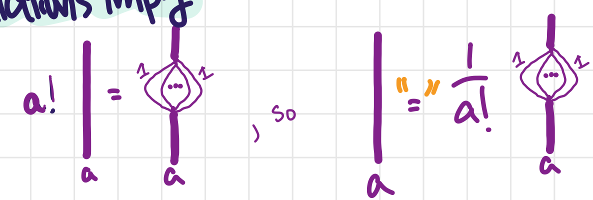
Super · Interchange Law:



· Coupons, e.g.

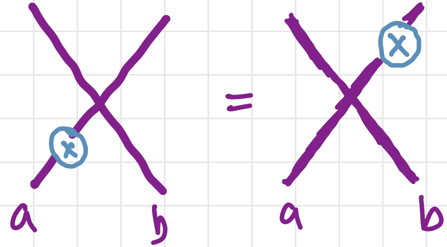


Relations Imply

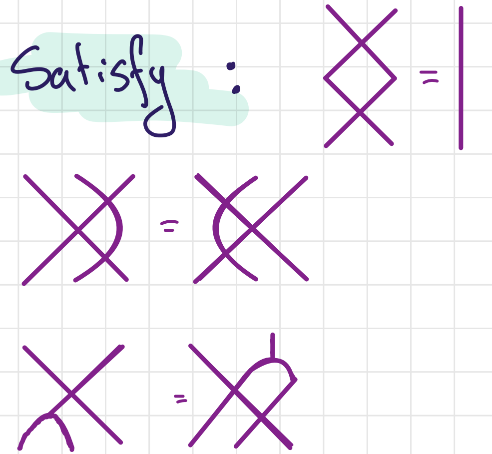


Last Relation ⤵

- Coupons slide past crossings

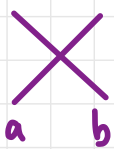
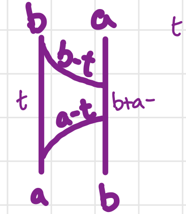


Satisfy:

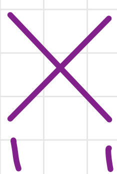
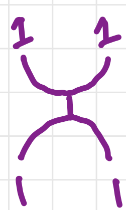


Crossings

For $a,b \geq 0$, define the crossing $(a,b) \to (b,a)$

by  $:= \sum_{t=0}^{\infty} (-1)^t$ 

Examples: $a=b=1$

 $=$  $-$ 

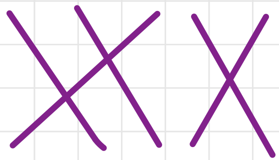
$t=0$ $\quad$ $t=1$

Thm: (DKMZ) For $d \geq 1$, there is an isomorphism:

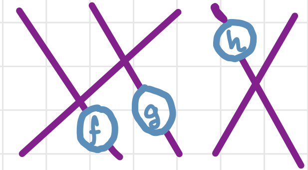
$$\mathrm{End}_{\mathrm{Web}_A}(1^d) \longrightarrow G_d \wr A$$

wreathe product algebra

Some Perspective



Symmetric Group G_d $\xrightarrow{\text{Superalg Def.}}$



Wreathe Product $G_d \wr A$

$\downarrow$ thicken

$\downarrow$ Thicken

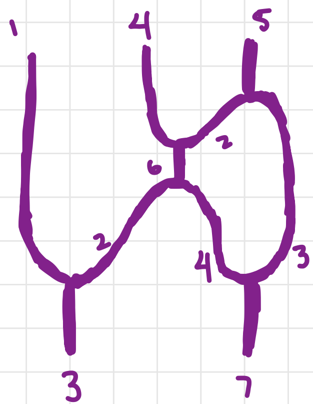
$\mathrm{Web}_{\Bbbk}$

Cautis-Kamnitzer-Morrison
Brundan-Entova-Aizenbud-Etingof-Ostrik

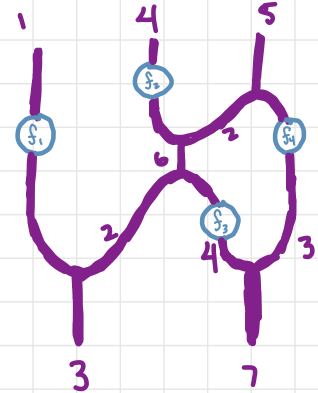
Web_A

thin strands

A = Clifford Alg gives q-Webs of Brown-Kujawa



Superalg Def.



Theorem (D.-Kujawa-Muth-Zhu, '23) For every $n \geq 1$, there is an asymptotically faithful, monoidal superfunctor $\Upsilon_n : \mathbf{Web}_A \longrightarrow \text{mod-}\mathfrak{gl}_n(A)_{\mathcal{S}}$ given by

objects

$\vec{a} \longrightarrow S^{\vec{a}} = S^{a_1} \otimes \cdots \otimes S^{a_k}$

$S^a = S^a(A^{\oplus n})$ (super) symmetric power of natural $\mathfrak{gl}_n(A)$ module

morphisms

$\longrightarrow$ merge : $S^a \otimes S^b \to S^{a+b}$

$\longrightarrow$ split : $S^{a+b} \to S^a \otimes S^b$

$\longrightarrow$ Left multiplication by x

$S^a \longrightarrow S^a$

Theorem (D.-Kujawa-Muth-Zhu, '23) For every $n \geq 1$, there is an asymptotically faithful, monoidal superfunctor $\Upsilon_n : \mathrm{Web}_A \longrightarrow \text{mod-}\mathfrak{gl}_n(A)$

- Asymptotic Faithfulness is used to prove a $\Bbbk$-basis Theorem for morphism spaces in Web_A
- We prove fullness for A = ss superalgebra, $\Bbbk$ = alg. closed field of char 0, but open outside of this setting.

(Previously known when $A = \Bbbk$ and A = Clifford algebra)

Schurifications

- For a superalgebra A, and $n, d \geq 1$, Kleshchev-Muth define the $\mathbb{k}$-superalgebras:

$$T^A_{\mathfrak{a}}(n,d) \subseteq \left(M_n(A)^{\otimes d}\right)^{\mathfrak{S}_d} \leftarrow \text{signed action}$$

subalgebra, full rank sublattice depending on choice of a subalgebra $\mathfrak{a} \subseteq A_{\bar{0}}$

- Evseev-Kleshchev: If $\mathbb{F}$ has char p, $A = Z_{p-1}$, $\mathfrak{a} = \mathbb{Z}_{p-1}$ = paths of length 0,

$$T^{Z_{p-1}}_{\mathfrak{z}_{p-1}}(d,d) \underset{\mathbb{F}}{\overset{\sim}{=}} \text{Weight } d \text{ RoCK}$$

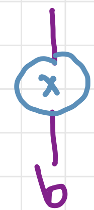
Morita

block for Symmetric group.

Schurification & Webs

Theorem (D-K-M-Z, '23)

Fix some subalgebra $\mathfrak{a} \leq A_{\bar{0}}$, and define $\mathbf{Web}^A_{\mathfrak{a}}$ as the subcategory of $\mathbf{Web}^A$ which only allows

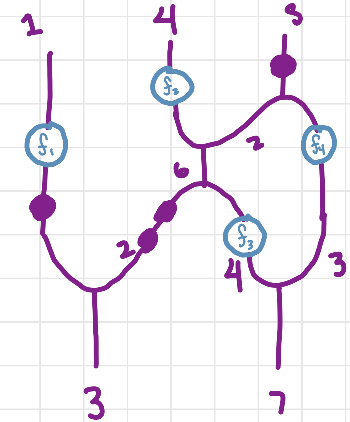


for $x \in \mathfrak{a}$ whenever $b > 1$.

Then, $$\bigoplus_{\vec{a},\vec{b} \in \Omega(n,d)} \mathrm{Hom}_{\mathbf{Web}^A_{\mathfrak{a}}}(\vec{a},\vec{b}) \cong T^A_{\mathfrak{a}}(n,d).$$

$\Omega(n,d)$ = compositions of n w/ d parts.

In Progress : When A is Frobenius, define **Affine Webs and Cyclotomic Quotients**



- "Thick" analogs of
  - degenerate affine Hecke alg
  - affine Wreath product algebras

- Expect applications to
  - Imaginary KLR
  - Rock Blocks for Symmetric group / Ariki-Koike
  - Spin RoCK blocks for sym. group (Kleshchev-Livesey)