NEARLY TORIC SCHUBERT VARIETIES

Mahir Bilen Can (based on joint works with Pinaki Saha and Nestor Diaz)

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Goal: Organizing algebraic group actions by tracking invariants of their restrictions to some important subgroups.

Main tool: Modality.

Context: Schubert and BSDH varieties.

The generic modality of an algebraic group action $G \times X \to X$ (X is irreducible) is defined by

$$d_G(X) := \mathrm{tr.deg}\; k(X)^G.$$

Rosenlicht: $d_G(X)$ is equal to the minimum codimension of a G-orbit in X.

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The modality of $G \times X \to X$ is defined by

$$\operatorname{mod}(G : X) = \max_{\substack{Y \subseteq X \\ Y:\ \text{irr., } G\text{-stable}}} d_G(Y).$$

Popov-Vinberg: $\operatorname{mod}(G : X) = 0$ iff G has only a finite number of orbits.

Example

The left multiplication action $GL_2(\mathbb{C})$ on $Mat_2(\mathbb{C})$ has an open orbit. Hence, we have

$$d_{GL_2(\mathbb{C})}(Mat_2(\mathbb{C})) = 0.$$

Note that $Y := Mat_2(\mathbb{C}) \setminus GL_2(\mathbb{C})$ is an irreducible 3-fold. The restriction of the action $GL_2(\mathbb{C}) \times Y \to Y$ has infinitely many maximal dimensional orbits but the minimum codimension of a $GL_2(\mathbb{C})$-orbit is 1. For example, we have the 2 dimensional orbit

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \qquad (ad - bc \neq 0).$$

In other words, we have $d_{GL_2(\mathbb{C})}(Y) = 1$. It follows that

$$\mathrm{mod}(GL_2(\mathbb{C}) : Mat_2(\mathbb{C})) = 1.$$

Let G be a connected reductive group. Let $G \times X \to X$ be an algebraic group action, where X is a normal variety.

Definition

The complexity of $G \times X \to X$ is defined by

$$c_G(X) := d_B(X),$$

where $B \subset G$ is a Borel subgroup. If $c_G(X) = 0$ holds, then X is called a spherical G-variety.

The condition $c_G(X) = 0$ is equivalent to B having an open orbit in X.

Theorem (Brion, Vinberg)

X is a spherical G-variety iff B has only finitely many orbit in X.

In summary, if X is a normal G-variety, where G is a connected reductive group, then the following are equivalent:

1. $c_G(X) = 0$.
2. $k(X)^B = k$.
3. $\mathrm{mod}(B : X) = 0$.
4. X has only finitely many B-orbits.
5. If X is quasi-projective, then for every G-linearizable line bundle $L \to X$, the G-module $H^0(X, L)$ is multiplicity-free.
6. If X is affine, then $k[X]$ is multiplicity-free.

Here is the simplest general example.

Example

Every toric variety is a spherical T-variety for some torus T.

Now we can introduce the family of varieties that we are interested in.

Definition

Let X be a normal G-variety, where G is a connected reductive group. Let $T \subset B$ be the maximal torus and a Borel subgroup of G. We say that X is a nearly toric G-variety if the following two conditions hold:

$$c_T(X) = 1 \quad \text{and} \quad c_G(X) = 0.$$

Example

Let X denote the space of degenerate 4×4 skew-symmetric matrices. Then

$$X \cong \bigwedge^2 \mathbb{C}^4 \setminus GL_4(\mathbb{C}) \cdot v,$$

where $v \in \bigwedge^2 \mathbb{C}^4$ is a 2-form in general position. It is well-known that the following action is spherical:

$$\begin{aligned} GL_4(\mathbb{C}) \times X &\longrightarrow X \\ (A, B) &\mapsto ABA^\top \end{aligned}$$

It is also easy to see that the restriction of the action of $GL_4(\mathbb{C})$ to its maximal torus T has (maximal) 4 dimensional orbits. Since $\dim X = 5$, we see that

$$c_T(X) = d_T(X) = 5 - 4 = 1.$$

Therefore, X is a nearly toric $GL_4(\mathbb{C})$-variety.

Standard notation:

- G : connected reductive group
- B : a Borel subgroup of G
- P : a standard parabolic subgroup of G
- T : maximal diagonal subgroup of B
- (W, S) : the Coxeter system of (G, B, T)
- W_P : Weyl group of (L, T), where $P = L \ltimes \mathscr{R}_u(P)$ and $T \subset L$
- W^P : minimal left coset representatives of W_P in W
- $\ell : W \to \mathbb{N}$: the length function
- G/B : flag variety
- G/P : partial flag variety

Definition

The T-fixed points of G/P are indexed by W^P. For $w \in W^P$, the Zariski closure

$$X_{wP} := \overline{B\dot{w}P/P} \subset G/P$$

is called a Schubert variety in G/P.

Schubert varieties are finite unions of B-orbits:

$$X_{wP} = \bigsqcup_{v \leq w \text{ in } W^P} B\dot{v}P/P,$$

where $\leq$ is the Bruhat-Chevalley order on W^P.

Thus $Stab_G(X_{wP})$ is a parabolic subgroup $Q \subset G$, and Levi factors of Q are the maximal reductive subgroups of G that act on X_{wP}.

Question

Is there a characterization of $w \in W$ such that X_{wB} is a toric variety (w.r.t. T)?

Let $w \in W$.

- w is called a Coxeter element if it is a product of all elements of S in some order without repetition.
- w is called a Coxeter-like element if it is a product of some elements of S in some order without repetition.

Theorem (Karuppuchamy)

With respect to T-action, X_{wB} is a toric variety if and only if w is a Coxeter element.

Said differently, $c_T(X_{wB}) = 0$ if and only if w is a Coxeter element.

1. Is there a characterization of $w \in W$ such that $c_T(X_{wB}) = 1$?
2. Is there a characterization of $w \in W$ such that $c_L(X_{wB}) = 0$?
3. Is there a characterization of $w \in W$ such that X_{wB} is a nearly toric Schubert variety?
4. If $A \subset W$ is a particular subset, is there a good* answer for $w \in A$?

* = combinatorial

In type A, there are explicit answers for the first three of these questions. For the last one, which can be regarded as a general combinatorial research area, there are some interesting families to consider.

In type A:

1. Is there a characterization of $w \in W$ such that $c_T(X_{wB}) = 1$?
   Answered: Lee-Park-Masuda (2021)
2. Is there a characterization of $w \in W$ such that $c_L(X_{wB}) = 0$?
   Answered: Gaetz (2022) - proving the conjecture of Gao-Hodges-Yong
3. Is there a characterization of $w \in W$ such that X_{wB} is a nearly toric Schubert variety?
   Answer: Can-Diaz (2023)
4. If $A \subset W$ is a particular subset, are there good answers to our previous questions for $w \in A$?
   Partially answered: Can-Diaz (2023)

In all types, Question 2 has been recently (April 2023) answered by Gao-Hodges-Yong and Can-Saha.

Here is a sample answer for Question 4.

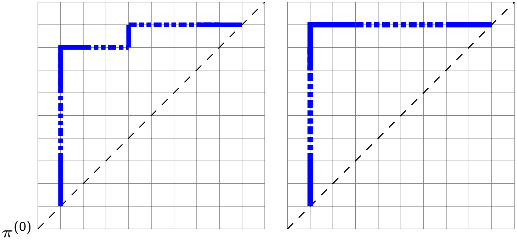
Let S_n denote the symmetric group on $\{1, \ldots, n\}$.

Theorem (Can-Diaz)

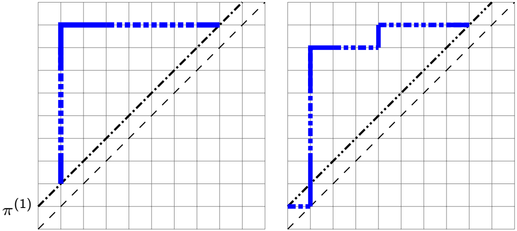
Let $A \subset S_n$ denote the set 312-avoding permutations. For $w \in A$, let π denote the corresponding Dyck path. Then X_{wB} is a spherical Schubert variety if and only if π is a spherical Dyck path.

Here, we call a Dyck path π a spherical Dyck path if

- every connected component of π on the first diagonal is either an elbow or a ledge, or
- every connected component of π on the second diagonal is an elbow, or a ledge whose E extension is the initial step of a connected component of π on the first diagonal.



(a) A ledge or an elbow of $\pi^{(0)}$.



(b) An elbow or a ledge of $\pi^{(1)}$.

Figure: Spherical Dyck paths

Here we describe our solution to Question 3. Let S_n denote the symmetric group on $\{1, \ldots, n\}$.

Theorem (Can-Diaz)

The Schubert varieties $X_{wB} \subset GL_n/B$ ($w \in S_n$) which are nearly toric are characterized by the following properties:

1. *If X_{wB} is singular, then $w \in S_n$ contains the pattern 3412 exactly once and avoids the pattern 321.*
2. *If X_{wB} is smooth, then w contains the pattern 321 exactly once and avoids the following patterns:*

$$\mathscr{P} := \left\{ \begin{matrix} 24531 & 25314 & 25341 & 34521 & 35421 \\ 42531 & 52314 & 52341 & 54213 & 54231 \\ 53124 & 53142 & 53421 & 54123 & 3412 \end{matrix} \right\}.$$

Needless to say here the proof of this theorem is built on the works of Lee-Masuda-Park and Gaetz.

Let $\underline{w} = (s_{i_1}, \dots, s_{i_m})$ be a word from S. Let $P_{i_j} := B \cup Bs_{i_j}B$ for $j \in \{1, \dots, m\}$.

Definition

The BSDH-variety $X_{\underline{w}}$ is the quotient of $P_{i_1} \times \cdots \times P_{i_m}$ by the following right action of B^m:

$$(p_1, \dots, p_m) \cdot (b_1, \dots, b_m) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{m-1}^{-1} p_m b_m).$$

If $w = s_{i_1} \cdots s_{i_m}$, then the Schubert variety X_{wB} is given by the image of

$$\begin{aligned} \mathbf{m}: \ X_{\underline{w}} &\longrightarrow G/B \\ [p_1, \dots, p_m] &\longmapsto p_1 \cdots p_m B. \end{aligned}$$

If $\underline{w}$ is a reduced word, then $\mathbf{m} : X_{\underline{w}} \to X_{wB}$ is a resolution of singularities.

Let $\underline{w} := (s_{i_1}, \ldots, s_{i_r})$ be a word in S. We call $G \times_B X_{wB}$ is a G-Schubert variety, and $G \times_B X_{\underline{w}}$ a G-BSDH variety.

If $\underline{w}$ is a reduced word, $1 \times \mathbf{m} : G \times_B X_{\underline{w}} \to G \times_B X_{wB}$ is a G-equivariant resolution of singularities.

Question

Let $X := G \times_B Z$ for $Z \in \{X_{wB}, X_{\underline{w}}\}$.

1. *Under what conditions X is a spherical G-variety?*
2. *Under what conditions X is a wonderful variety?*
3. *If X is not a spherical G-variety, then does it possess any pleasant properties at all?*

We begin answering our question from last to first.

Proposition (Can-Saha)

Let $\underline{w}$ be a word in S. Let r be a nonnegative integer. Then we have

$$mod(G : G \times_B X_{\underline{w}}) = r \iff mod(B : X_{\underline{w}}) = r.$$

In particular, if $\underline{w}$ is a reduced word of length l, then we have $mod(B : X_{\underline{v}}) = 0$ for every subword $\underline{v}$ of length $l-1$ if and only if we have $mod(G : G \times_B X_{\underline{w}}) = 0$.

Theorem (Can-Saha)

Let $\underline{w}$ be a word in S. Let $w \in W$ denote the associated element of W. Let X denote either $G \times_B X_{\underline{w}}$ that is a G-BSDH variety or $G \times_B X_{wB}$, that is a G-Schubert variety. If a B-stable divisor D in X contains a G-orbit, then D is G-stable. In other words, X always behaves like a spherical toroidal G-variety.

To answer the questions 1 and 2, we make use of the works of Avdeev, Luna, and Karuppuchamy.

Theorem (Can-Saha)

Let $\underline{w}$ be a reduced word in S. Then the following statements are equivalent:

1. $X_{\underline{w}}$ *is a toric variety.*
2. X_{wB} *is a toric variety.*
3. $G \times_B X_{\underline{w}}$ *is a spherical G-variety.*
4. $G \times_B X_{wB}$ *is a spherical G-variety.*

Furthermore, $G \times_B X_{\underline{w}}$ *is a wonderful variety iff* X_{wB} *is a toric variety.*

Let $X_{wB} \subset G/B$. The stabilizer of X_{wB} in G is the standard parabolic subgroup $P_{J(w)}$ generated by B and $J(w) = \{s \in S : \ell(sw) < \ell(w)\}$. We set

$$L(w) := \text{standard Levi factor of } P_{J(w)}, \quad W_{J(w)} := \text{the Weyl group of } L(w).$$

Theorem (Can-Saha)

Let $w \in W$. Then the associated Schubert variety X_{wB} is a spherical $L(w)$-variety such that $\dim B_{L(w)} = \dim X_{wB}$ if and only if w can be written as

$$w = w_{0,J(w)} c,$$

where $w_{0,J(w)}$ is the longest element of $W_{J(w)}$ and c is a Coxeter element of W such that $\ell(w) = \ell(w_{0,J(w)}) + \ell(c)$.

Remark

The statement of our theorem was conjectured by Gao, Hodges and Yong not so long ago. After proving our theorem, we learned that they proved their conjecture at the same time as us.

We extend our previous result to the BSDH-varieties.

Let $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_r})$ be a reduced word, let $(\alpha_{i_1}, \dots, \alpha_{i_r})$ denote the corresponding sequence of simple roots. Define $J(\underline{w})$ as the set of simple roots α_{i_j} from the list $(\alpha_{i_1}, \dots, \alpha_{i_r})$ such that all of the simple roots α_{i_k} with $1 \leq k \leq j$ commute with s_{i_j}. Define $L(\underline{w})$ as the Levi determined by $J(\underline{w})$.

Theorem (Can-Saha)

Let $\underline{w}$ be a reduced word. Then $X_{\underline{w}}$ is a spherical $L(\underline{w})$-variety if and only if $w_{0,J(\underline{w})}w$ is a product of distinct simple reflections, where w is the element of W associated with $\underline{w}$ and $w_{0,J(\underline{w})}$ denotes the longest element of $W_{J(\underline{w})}$.

Corollary (Can-Saha)

Let $\underline{w}$ be a reduced word. Then, $X_{\underline{w}}$ is a spherical $L(\underline{s_{i_1}})$-variety if and only if $s_{i_1}w$ is a product of distinct simple reflections.

Another surprising application of the sphericality is on the singularities of Schubert varieties.

Theorem (Can-Saha)

Let $w \in W$. Let J be a subset of $J(w)$. Let L_J denote the corresponding standard Levi factor. We assume that X_{wB} is a spherical L_J-variety such that $\dim X_{wB} = \dim B_{L_J}$. Then the following are equivalent:

1. *X_{wB} is a (rationally) smooth Schubert variety in G/B,*
2. *$X_{c^{-1}P_J}$ is a (rationally) smooth toric variety in G/P_J,*
3. *$|\{r \in R : \ y \leq ry \leq vw_{0,J}\}| = \ell(w_{0,J}) + \ell(v) - \ell(y)$ for all $y \in [w_{0,J}, vw_{0,J}]$, where R is the union of all conjugates of S in W.*

Part 3 of our theorem is where we used KL theory based on the works of Carrell and Deodhar.

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Part 3 of our theorem is where we used KL theory based on the works of Carrell and Deodhar.

... AND THIS WAS THE END OF OUR TALK!