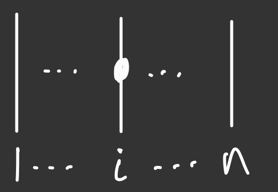
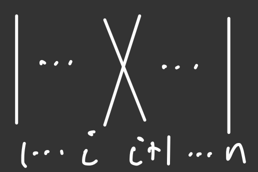
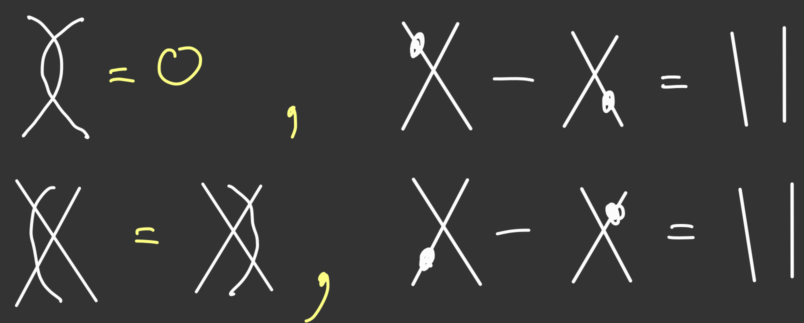
Reminders about the nil-Hecke algebra

Work over a field $\Bbbk$, later char $\Bbbk \neq 2$

NH_n gens $\underbrace{x_1,\dots,x_n}_{\text{poly. rels}}$, $\underbrace{\tau_1,\dots,\tau_{n-1}}_{\text{braid rels}}$ + $\left.\begin{array}{l}\tau_i^2 = 0, \quad x_i\tau_i = \tau_i x_{i+1} + 1 \\ \tau_i x_i = x_{i+1}\tau_i + 1\end{array}\right\}$

Will use diagrams! $x_i =$ 

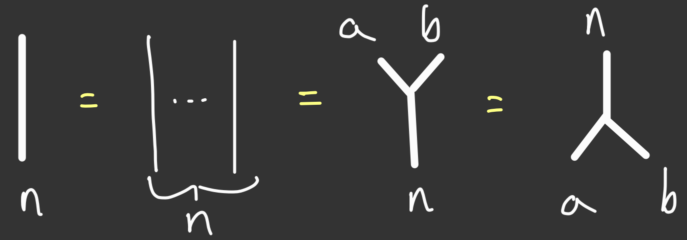
$\tau_i =$ 

Locally: 

(plus "interchange" if far apart)

More tricks:

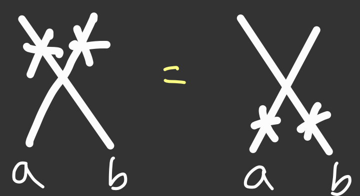
$a+b=n$



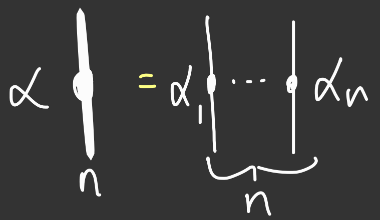
all are 1_n identity

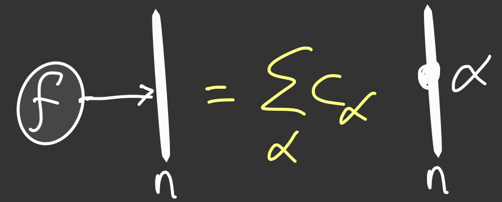
thick crossing

= longest element =



etc...

<u>Dots</u> $\alpha \in \mathbb{N}^n$: 

<u>Pins</u> $f = \sum_\alpha c_\alpha x^\alpha \in \Bbbk[x_1,\dots,x_n]$: 

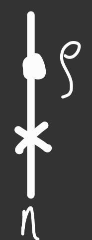
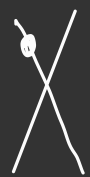
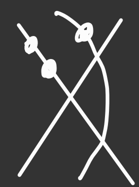
Will drop subscript n when on string of thickness n

<u>Special cases</u> $\alpha = \rho_n = (n-1, n-2, \dots, 1, 0)$

$\alpha = \omega_{r,n} = (\underbrace{1,\dots,1}_{r}, 0, \dots, 0)$ <u>fundamental weight</u>

$f = e_{r,n} = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r}$: <u>elementary symmetric polynomial</u> in $x_1, \dots, x_n$

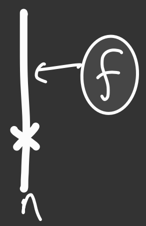
$f = s_{\lambda,n}$: <u>Schur polynomial</u> in $x_1, \dots, x_n$ indexed by partition λ, $\ell(\lambda) \leq n$

eg  $n=1$  $n=2$  $n=3$  etc...

It is well known that this is a <u>primitive</u> homogeneous <u>idempotent</u> in NH_n (and all such are conjugate to this one)

$\begin{cases} x_i : \text{degree } 2 \\ \tau_i : \text{degree } -2 \end{cases}$

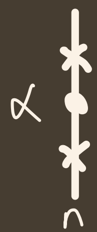
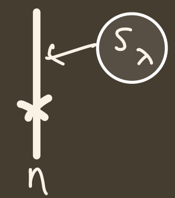
$NH_n \circlearrowright \Bbbk[x_1,\dots,x_n]$ x_i acts by mult., τ_i acts by Demazure $\tau_i \cdot f = \dfrac{s_i(f)-f}{x_{i+1}-x_i}$

$\overset{\in}{f} \longleftrightarrow q^{-n(n-1)}$  the left action becomes action "on top" of diagram

Also $\Lambda_n = \Bbbk[x_1,\dots,x_n]^{S_n} = Z(NH_n)$ and $\Bbbk[x_1,\dots,x_n]$ is free Λ_n-module, basis x^α $(0 \le \alpha_i \le n-i)$

$\Rightarrow NH_n \xrightarrow{\sim} \mathrm{End}_{\Lambda_n}(\Bbbk[x_1,\dots,x_n]) \cong \mathrm{Mat}_{n!}(\Lambda_n)$

<u>Fundamental theorem of Schubert calculus</u>

 $= \begin{cases} (-1)^{\ell(w)} & \text{if } w(\alpha+\rho) = \lambda+\rho \text{ for } w \in S_n \text{ and partition } \lambda \\ 0 & \text{else.} \end{cases}$ 

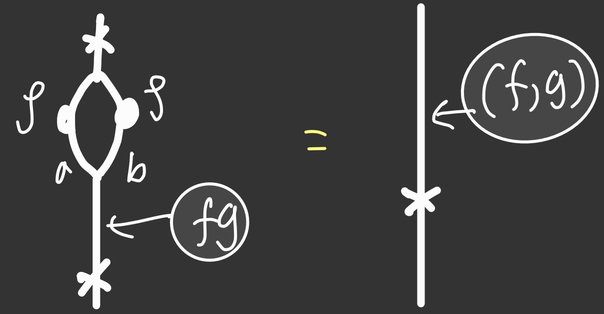
You deduce that  acts as 1 on x^ρ, 0 on other x^α $(0 \le \alpha_i \le n-i)$

$\Rightarrow$ earlier claims about this primitive idempotent

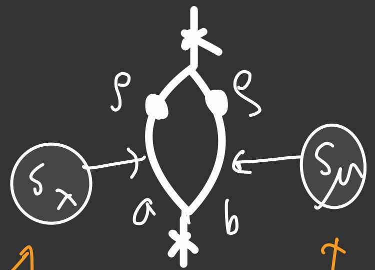
$\Lambda_{a,b} := \mathbb{k}[x_1,\dots,x_n]^{S_a \times S_b}$ is equivariant cohomology of Gr^n_a $(n = a+b)$

Λ_n Frobenius extension

$\exists$ non-deg. symm. Λ_n-bilinear form $(\cdot,\cdot) : \Lambda_{a,b} \times \Lambda_{a,b} \to \Lambda_n$



Theorem implies



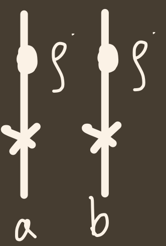
$= \begin{cases} \mathrm{sgn}(\lambda) \text{ [strand with star] } & \text{if } \mu = \tilde{\lambda} \\ 0 & \text{else} \end{cases}$

λ fits in $a \times b$ rectangle

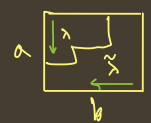
μ in $b \times a$ rectangle

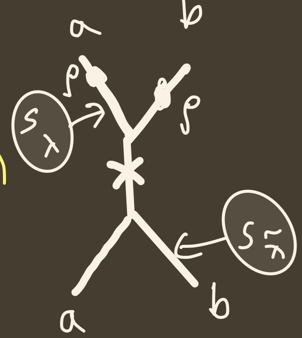
$\Rightarrow$ required dual bases for $\Lambda_{a,b}$ as free Λ_n-module (Schubert classes!)

Corollary (KLMS)



$= \sum_{\lambda} \mathrm{sgn}(\lambda)$





as a sum of mutually orthogonal, primitive homogeneous idempotents.

Nil-Brauer and the $\imath$-quantum group of rank 1

I use $*$ for tensor / horizontal comp.

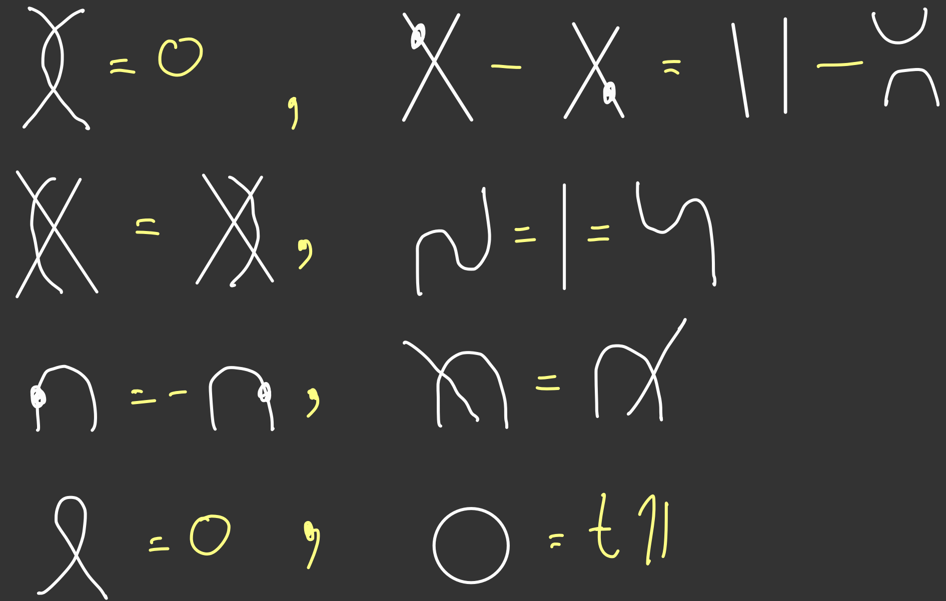
$\mathcal{NB}_t$ strict $\Bbbk$-linear graded monoidal category, defined by generators & relations

Generating object B, $1_B = |$ (obs $B^{*n} \leftrightarrow \mathbb{N}$)

Generating morphisms 

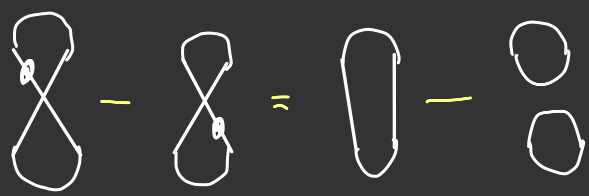
deg 2 deg −2 deg 0

Relations



Here $t \in \Bbbk$ is parameter.

Actually $t=0$ or $t=1$ else trivial!

Proof: 

 $\therefore\ t^2 1\!\!1 = t\, 1\!\!1 \quad \therefore\ t=0$ or $t=1$

What's $\mathrm{End}_{\mathcal{NB}_t}(1\!\!1)$? Have dotted bubbles 

Recall Schur's q-functions $q_r = \sum_{s=0}^{r} e_s h_{r-s} \in \Lambda$ (algebra of symmetric functions)

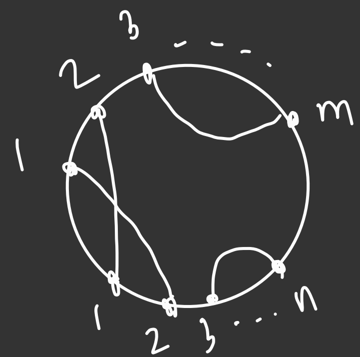
They generate subalgebra $\Gamma = \Bbbk[q_1, q_3, q_5, \dots]$ of Λ.

Theorem (B.-Wang-Webster) $\Gamma \xrightarrow{\sim} \mathrm{End}_{\mathcal{NB}_t}(\mathbb{1})$, $q_r \mapsto -2(-1)^r \bigcirc^{\bullet\, r}$

Each morphism space $\mathrm{Hom}_{\mathcal{NB}_t}(n,m)$ is free as a Γ-module with "obvious" basis given by reduced Brauer diagrams with dots at fixed point on each string.

$$\Rightarrow \mathrm{rank}_\Gamma \mathrm{Hom}_{\mathcal{NB}_t}(n,m) = \frac{1}{(1-q^{-2})^{\frac{m+n}{2}}} \sum_{\substack{\text{chord diagrams } C \text{ with} \\ m+n \text{ marked boundary points}}} q^{2\#\{\text{crossings in } C\}}$$

($m \equiv n$ (2) of course)



which by some combinatorics equals $(B^n, B^m)^\imath$

in the $\imath$-quantum group $U^\imath$ of rank one!

$U^\imath$ = subalgebra of usual $U_q(\mathfrak{sl}_2)$ generated by $B = F + qK^{-1}E$

$(u,v)^\imath = \lim_{\lambda\to\infty} (u\eta_\lambda, v\eta_\lambda)$ ← usual form on $U_q(\mathfrak{sl}_2)$-module of h/w $\lambda \in \mathbb{N}$, η_λ = h/w vec.

in $\mathbb{Q}((q^{-1}))$

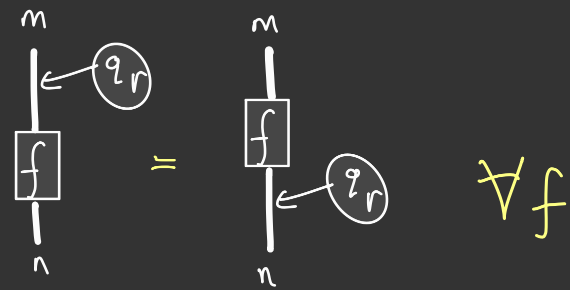
introduced by Bao-Wang

Primitive idempotents in NB_t

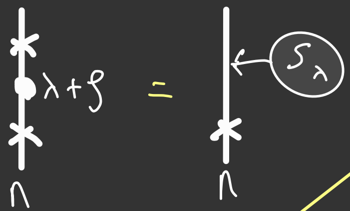
NB_t has a lot in common with nil-Hecke, but

much harder!

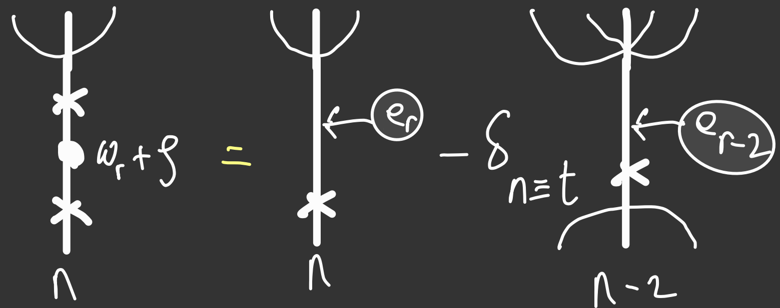
It is no longer true that all symmetric polynomials are central but:



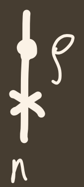
Also false:



But remarkably still have

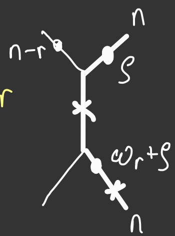


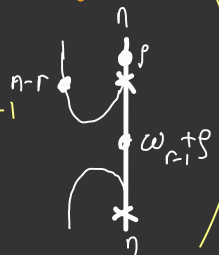
← key to next theorem!

Theorem (BWW) $e_n :=$ 

is a primitive homogeneous idempotent in $\mathcal{NB}_t$, and every such is conjugate to one of these. Moreover we decompose $B \star e_n$ as a sum of conjugates of these.

If $n \not\equiv t$ (2) we show

$B \star e_n = \sum_{r=0}^{n} \Big((-1)^r$ 

$+ (-1)^{r-1}$ 

$\Big)$

for $r \geq 1$ only

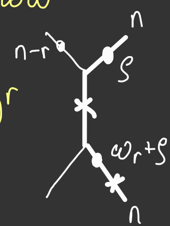
$\wr$

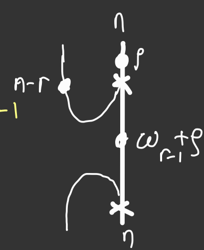
e_{n+1}

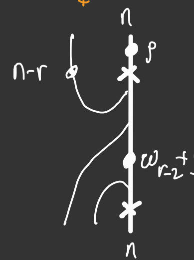
($n+1$ of these)

sums of mut. orthogonal primitive homogeneous idemps.

If $n \equiv t$ (2) we show

$B \star e_n = \sum_{r=0}^{n} (-1)^r$ 

$+ \sum_{r=1}^{n} \Big((-1)^{r-1}$ 

$+ (-1)^r$ 

$\Big)$

for $r \geq 2$ only

$\wr$

e_{n+1}

($n+1$ of these)

$\wr$

e_{n-1}

(n of these)

Now consider modules over $\mathcal{NB}_t$ (graded $\Bbbk$-linear functors to $\mathcal{G}\mathrm{Vec}$).

Traditional to pass from category to its <u>path algebra</u> $NB = \bigoplus_{m,n \geq 0} \underbrace{\mathrm{Hom}_{\mathcal{NB}_t}(n,m)}_{1_m NB 1_n}$.

This is a locally unital graded algebra with system 1_n $(n \in \mathbb{N})$ of mutually orthogonal idempotents.

$K_0(NB)$ = Grothendieck $\mathbb{Z}[q,q^{-1}]$-algebra of f.g. graded projective left NB-modules

q acts by grading shift functor

$[P][Q] = [P \circledast Q]$ where $\circledast$ = induction product / Day convolution

Let $P(n) = q^{-\frac{1}{2}n(n-1)} NB\, e_n$ (projective assoc. to primitive idempotent e_n)

<u>Corollary</u> $B\,P(n) := P(1) \circledast P(n) = \begin{cases} [n+1]P(n+1) \oplus [n]\,P(n) & n \equiv t\ (2) \\ [n+1]\,P(n+1) & n \not\equiv t\ (2) \end{cases}$

The recurrence relation here is the same as recurrence for Bao-Wang's $\imath$-canonical basis $B_t^{(n)}$ $(n \in \mathbb{N})$ for $U^\imath$ (which also depends on choice of t).

In fact, this is how we prove e_n is primitive / $P(n)$ is indecomposable, for $(B_t^{(n)}, B_t^{(m)}) \equiv \delta_{n,m} \pmod{q^{-1}\mathbb{Q}[[q^{-1}]]}$ by the general theory.

So:

Theorem (BWW) $K_0(NB) \cong {}_{\mathbb{Z}}U_t^\imath$ as a $\mathbb{Z}[q,q^{-1}]$-algebra

$$[P(n)] \leftrightarrow B_t^{(n)}$$

where ${}_{\mathbb{Z}}U_t^\imath$ is the $\mathbb{Z}[q,q^{-1}]$-form generated by the $B_t^{(n)}$

i.e. NB_t categorifies the split $\imath$-quantum group of rank 1.

There's more interesting structure to NB — it has a graded triangular basis.

Sort of a weak triangular decomposition like in Lie theory.

Cantan role is played by $\bigoplus_{n\geq 0} NH_n \otimes_{\Bbbk} \Gamma$

good homological properties

Get standard modules $\Delta(n)$ $(n \in \mathbb{N})$, "affine highest weight category".

But $P(n)$ has a filtration

$\Delta(n)$
$\bigoplus q^{?} \Delta(n-2)$
$\vdots$
$\bigoplus q^{?} \Delta(1 \text{ or } 0)$

has submodules which are not finitely gen'd

layers are infinite direct sums (known mults.) which is a new phenomenon for me in the highest weight story

Kac-Moody 2-categories (categorifying usual quantum groups) also have this sort of structure (not so explicit).