Meta-Kazhdan–Lusztig combinatorics

joint work with M. De Visscher, A Hazi, E. Norton, N. Farrell, C. Stroppel

What Lie theoretic objects to we fully understand?

Take your favourite class of non-semisimple Lie theoretic objects:

- Category $\mathcal{O}$ for Lie algebras
- Polynomial representation of reductive groups
- Group algebras of Coxeter groups
- Finite-dimensional representations of supergroups
- (Quiver) Hecke algebras
- Quantum groups
- Khovanov arc algebras
- Anti-spherical Hecke categories. . .

What can you say about their structure?

- Characters of simples/composition factors of Verma modules? 

But what if we want more detail?

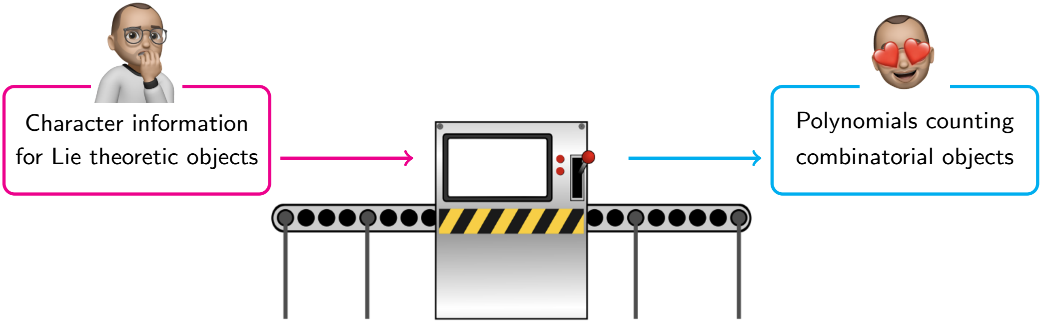
- Submodule structure of Verma modules? $\mathrm{SL}_2(\Bbbk)$ **and symmetric powers.**
- Gabriel-style presentations by Ext-quiver and relations? $\Bbbk\mathfrak{S}_p$ **and its Schur algebra.**

That's all?!

Section 1

Character theory and p-Kazhdan–Lusztig combinatorics

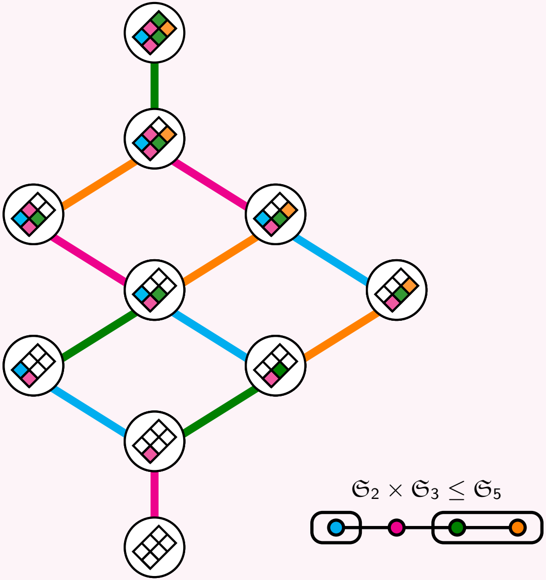
What is p-Kazhdan–Lusztig theory?



p-Kazhdan–Lusztig theory

Step 1: Label the simple/Verma modules

- We let $(\mathcal{P}_{m,n}, \leq)$ denote the poset of partitions which fit into an $(m \times n)$-rectangle ordered by inclusion. For example, let $m = 3$ and $n = 2$. Then



The poset $(\mathcal{P}_{m,n}, \leq)$ is important for many categories of interest...

Step 1: Label the simple/Verma modules

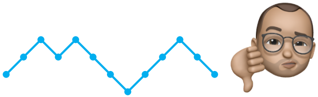
The simple and Verma modules of rational representations of supergroups $\mathrm{GL}_{m|n}(\mathbb{C})$, categories of perverse sheaves on Grassmannians, blocks of walled Brauer algebras, certain level 2 quiver Schur algebras, Khovanov arc algebras, and anti-spherical Hecke categories (of $\mathfrak{S}_m \times \mathfrak{S}_n \leq \mathfrak{S}_{m+n}$) are of the form

$$\{L(\lambda) \mid \lambda \in (\mathcal{P}_{m,n}, \leq)\} \qquad \{\Delta(\lambda) \mid \lambda \in (\mathcal{P}_{m,n}, \leq)\}$$

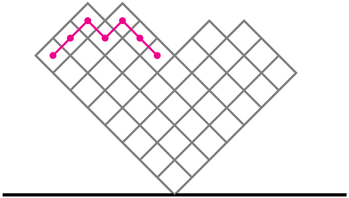
these categories are highest weight with respect to $\leq$.

Step 2: Define the Kazhdan–Lusztig polynomials of type $\mathfrak{S}_m \times \mathfrak{S}_n \leq \mathfrak{S}_{m+n}$

- A Dyck path is a path in NE/SE/NW/SW directions, finishing at the height at which it started, never dropping below this height:

- Let P be a Dyck path on the boxes of $\mu = (8^3, 7, 4^2, 3)$,



- We say that a pair of Dyck paths P and Q on μ are good if the rightmost box of P is not NW/SW of the leftmost box of Q.

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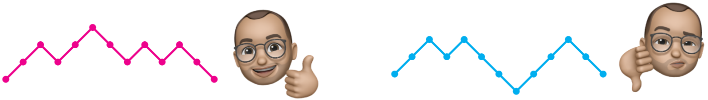

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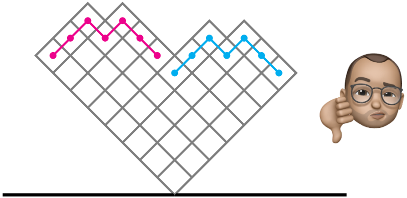

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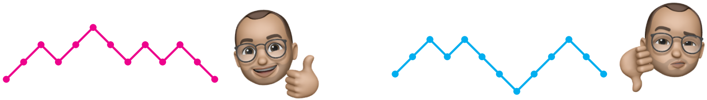

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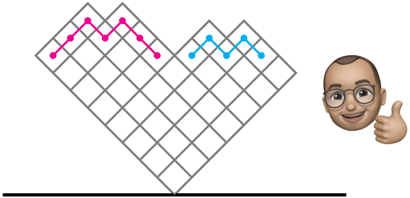

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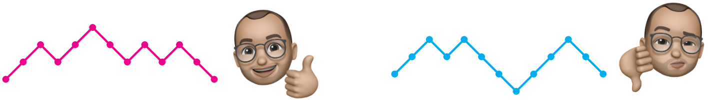

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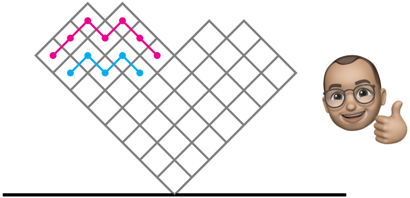

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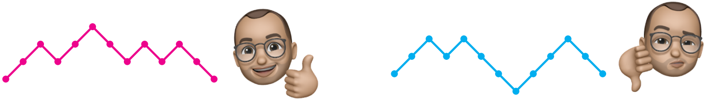

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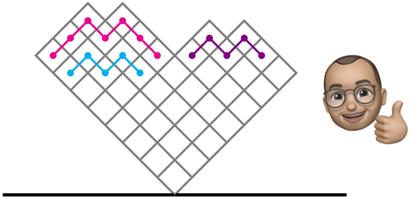

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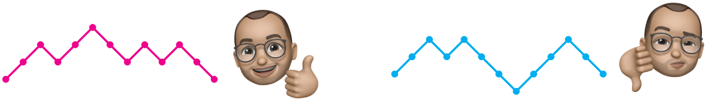

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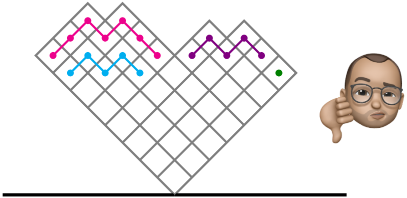

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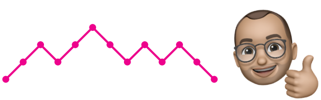
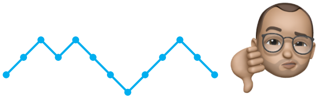

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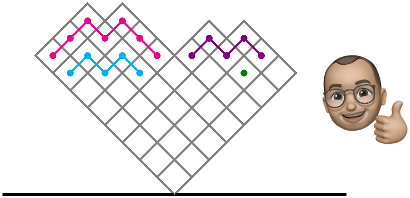

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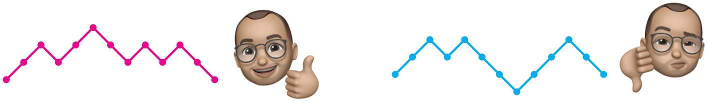

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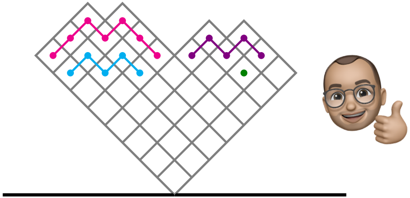
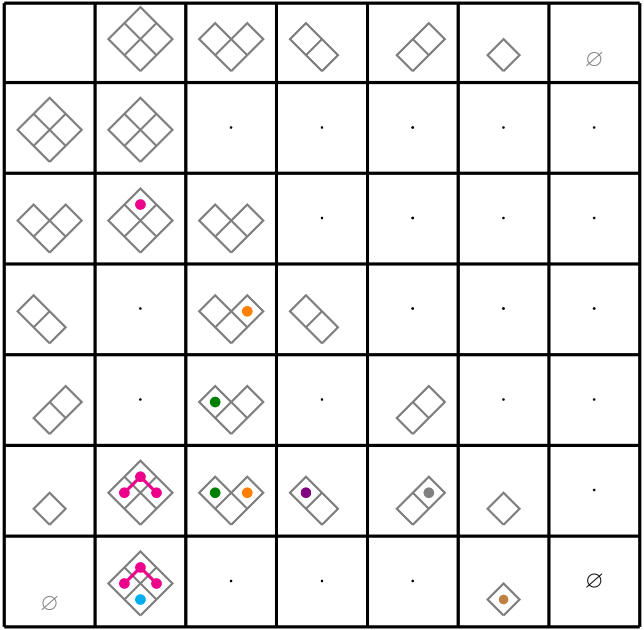


- We say that a pair of Dyck paths P and Q on μ are good if the rightmost box of P is not NW/SW of the leftmost box of Q.
- If $\lambda < \mu$ is such that $\mu - \lambda$ is tile-able by (pairwise) good Dyck paths, then we define

$$n_{\lambda,\mu} = q^{\sharp\{\text{good Dyck paths in tiling of } \mu-\lambda\}}$$

Otherwise we set $n_{\lambda,\mu} = 0$.

An example for $m = n = 2$



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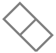
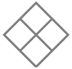
						$\emptyset$
	1	·	·	·	·	·
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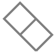
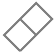
	·	q	1	·	·	·
	·	q	·	1	·	·
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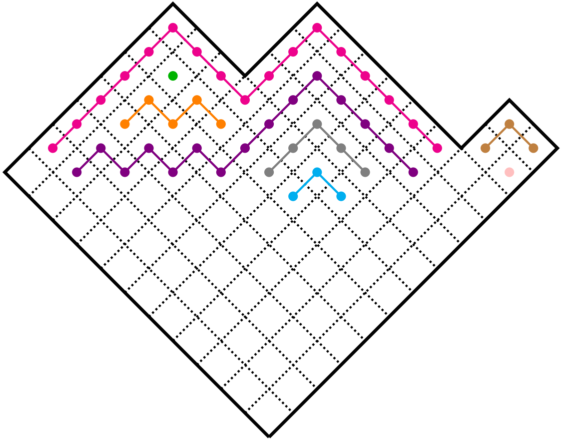
A bigger example

- Let $\mu = (11^7, 8^3, 2^2)$ and $\lambda = (11, 9, 8, 7, 6, 4, 3^2, 2^2)$.
- Any tiling of $\mu - \lambda$ has 8 Dyck paths in it and so

$$n_{\lambda,\mu} = q^8.$$

- There are 12 different Dyck tilings of $\mu - \lambda$. Here are some of them:



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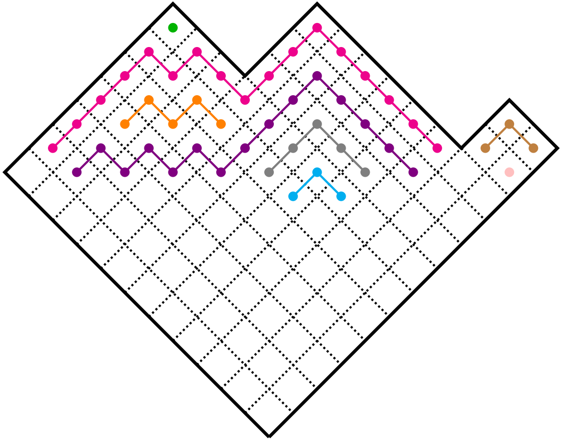

Important fact 1

The Kazhdan–Lusztig polynomials $\lambda, \mu \in \mathcal{P}_{m,n}$ are

$$n_{\lambda,\mu} = \begin{cases} q^{\sharp\{\text{good Dyck paths in tiling of } \mu-\lambda\}} & \text{if } \mu - \lambda \text{ is tile-able by good Dyck paths} \\ 0 & \text{otherwise} \end{cases}$$

Important fact 2

The decomposition numbers of rational representations of supergroups $\mathrm{GL}_{m|n}(\mathbb{C})$, categories of perverse sheaves on Grassmannians, blocks of walled Brauer algebras, certain level 2 quiver Schur algebras, Khovanov arc algebras, and anti-spherical Hecke categories (of $\mathfrak{S}_m \times \mathfrak{S}_n \leq \mathfrak{S}_{m+n}$) are of the form

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What are the limits to what Kazhdan–Lusztig combinatorics can tell us?

Instead of looking only at the sets of Dyck tableaux (which enumerate the p-Kazhdan–Lusztig polynomials) we want to look at the relationships for passing between these Dyck tableaux...

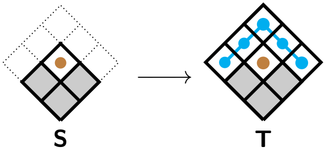
Section 2

Meta-Kazhdan–Lusztig combinatorics: Verma modules

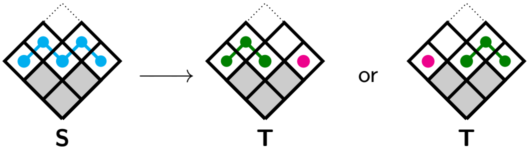
A partial ordering on Dyck tableaux of shape λ

Let **S** and **T** be two Dyck tableau of shape λ. Let $\deg(\mathbf{S}) = k$ and $\deg(\mathbf{T}) = k + 1$. We write $\mathbf{S} \to \mathbf{T}$ if either:

- $\mathbf{T} = \mathbf{S} \cup P$ for P an addable Dyck path.

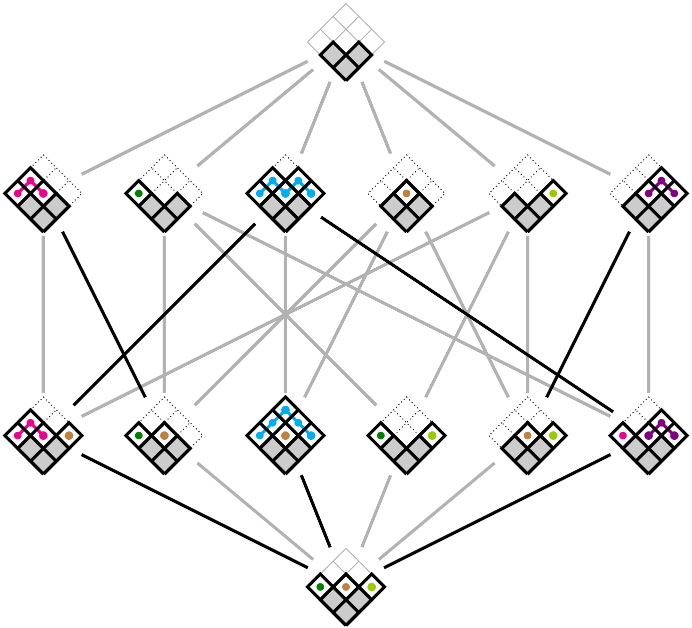


- **T** is obtained by splitting some $P \in \mathbf{S}$ into two distinct parts:



- We extend this to a partial ordering by transitivity.

The full submodule lattice of $\Delta_{3,3}(2,1)$



Theorem (Bowman De Visscher Hazi Stroppel)

Let $(W, P) = (\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$. The Alperin diagram for a Verma module for the anti-spherical Hecke category is given by the "add" and "split" operators on Dyck tableaux. Every Verma module has simple socle and is rigid.

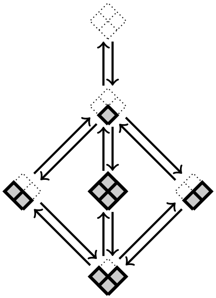
Section 3

Meta-Kazhdan–Lusztig combinatorics: Ext-quivers and relations

By a famous theorem of Gabriel, every algebra is Morita equivalent to the path algebra of its Ext-quiver modulo relations...

Theorem (Bowman De Visscher Hazi Stroppel)

The Ext-quiver is given by the combinatorially defined "Dyck quiver" $D_{m,n}$ with arrows corresponding to single Dyck paths. For example

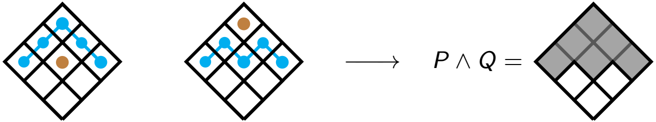


The basic algebra of the anti-spherical Hecke category for $(\mathfrak{S}_{m+n}, \mathfrak{S}_n \times \mathfrak{S}_m)$ is isomorphic to the quotient of the path algebra of $D_{m,n}$ modulo "Dyck relations"...

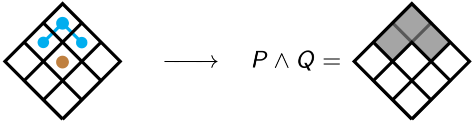
Cinching partitions

Let $P, Q \subseteq \mu$ be two distinct Dyck paths. We want to understand products $d_{\mu}^{\mu \pm P} d_{\mu \pm Q}^{\mu}$.

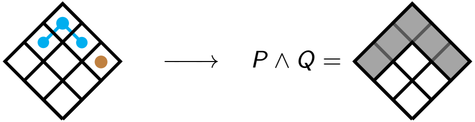
- If $P \sqcup Q$ admits two Dyck tilings, we set $P \wedge Q = P \sqcup Q$.



- If there is just one Dyck tiling of $P \sqcup Q$ and $b(P) < b(Q)$, then we set $P \wedge Q = Q$.



- If $P \sqcup Q$ is not a Dyck tiling, we set $P \wedge Q$ to be the smallest removable Dyck path of μ containing $P \sqcup Q$.



Theorem (Bowman De Visscher Hazi Stroppel)

The anti-spherical Hecke category for $(\mathfrak{S}_{m+n}, \mathfrak{S}_n \times \mathfrak{S}_m)$ is the associative $\mathbb{k}$-algebra generated by the elements

$$\{d_\mu^\lambda, d_\lambda^\mu \mid \lambda \subseteq \mu \text{ are a Dyck pair of degree } 1\} \cup \{1_\mu \mid \mu \in {}^P W\}$$

subject to the following relations and their duals. The idempotent relations

$$1_\mu 1_\lambda = \delta_{\lambda,\mu} 1_\lambda \qquad 1_\lambda d_\mu^\lambda 1_\mu = d_\mu^\lambda.$$

For $P \neq Q$ then we have that

$$d_\mu^{\mu-P} d_{\mu\pm Q}^\mu = (-1)^{sgn(P\wedge Q)} d_{\mu-P\wedge Q}^{\mu-P} d_{\mu\pm Q}^{\mu-P\wedge Q}$$

if $P \wedge Q$ is defined and 0 otherwise. For $P = \{[r,c], \dots, [r+b, c-b]\}$ we have

$$d_\mu^{\mu-P} d_{\mu-P}^\mu = \sum_{[r-1,c]\in Q} (-1)^{sgn(Q)} d_{\mu-P-Q}^{\mu-P} d_{\mu-P}^{\mu-P-Q} + \sum_{[r+b,c-b-1]\in Q} (-1)^{sgn(Q)} d_{\mu-P-Q}^{\mu-P} d_{\mu-P}^{\mu-P-Q}.$$

Summary I

We have complete understanding of the full algebra structure of the anti-spherical Hecke category for $(\mathfrak{S}_{m+n}, \mathfrak{S}_n \times \mathfrak{S}_m)$ in terms of explicit Dyck combinatorics.

Summary II

This presentation is characteristic-free and lifts to $\mathbb{Z}$.

Also I

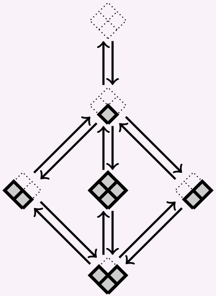
The anti-spherical Hecke category for $(\mathfrak{S}_{m+n}, \mathfrak{S}_n \times \mathfrak{S}_m)$ is isomorphic as a graded $\Bbbk$-algebra to the Khovanov arc algebra for $\Bbbk$ an arbitrary field.

Also II

We have complete understanding of the structure of rational representations of supergroups $\mathrm{GL}_{m|n}(\mathbb{C})$, categories of perverse sheaves on Grassmannians, blocks of walled Brauer algebras, and certain level 2 quiver Schur algebras.

An example $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$

The anti-spherical Hecke category of $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ is the path algebra of the quiver



modulo the following relations and their duals

$$d^{\varnothing}_{(1)} d^{(1)}_{(2)} = 0 = d^{\varnothing}_{(1)} d^{(1)}_{(1^2)} \qquad d^{(1)}_{(2)} d^{(2)}_{(2,1)} = d^{(1)}_{(2^2)} d^{(2^2)}_{(2,1)} = d^{(1)}_{(1^2)} d^{(1^2)}_{(2,1)} \qquad d^{(1)}_{\lambda} d^{\lambda}_{(1)} = -d^{(1)}_{\varnothing} d^{\varnothing}_{(1)}$$

$$d^{(2,1)}_{(2^2)} d^{(2^2)}_{(2,1)} = -d^{(2,1)}_{(2)} d^{(2)}_{(2,1)} - d^{(2,1)}_{(1^2)} d^{(1^2)}_{(2,1)}$$

for all λ and for any pair $\lambda < \mu$ not of the above form, we have that $d^{\lambda}_{\mu} d^{\mu}_{\lambda} = 0$.