LECTURE 12. SUSY GAUGE THEORIES. VECTOR SUPERFIELDS

1. Vector Superfield

Vector superfield is defined in terms of a constraint

$$V(x,\theta,\bar\theta)=V^\dagger(x,\theta,\bar\theta) \tag{1}$$

The superfield can be obtained by writing an expansion in terms of θ_α and $\bar\theta_{\dot\alpha}$ and imposing the reality condition

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x)+i\theta^\alpha\chi_\alpha(x)-i\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x)+\\ &+\frac{i}{2}\theta^\alpha\theta_\alpha(M(x)+iN(X))-\frac{i}{2}\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}(M(x)-iN(X))-\theta^\alpha\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}v_\mu(x)+\\ &+i(\theta^\alpha\theta_\alpha)\bar\theta_{\dot\alpha}[\bar\lambda^{\dot\alpha}(x)+\frac{i}{2}(\bar\sigma^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)]-i(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})\theta^\alpha[\lambda_\alpha(x)-\frac{i}{2}(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\bar\chi^{\dot\beta}(x)]\\ &+\frac{1}{2}(\theta^\alpha\theta_\alpha)(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})(D(x)+\frac{1}{2}\Box C(x)) \end{aligned} \tag{2}$$

The superfield is called "vector", because it contains a vector field $v_\mu(x)$. The bosonic fields $C(x), M(x), N(X), D(x)$ and $v_\mu(x)$ are real.

Supersymmetric gauge transformations are defined as

$$V\to V+\Phi+\Phi^\dagger \tag{3}$$

where Φ is a chiral superfield. Let us rewrite this equation in the component form

$$\delta C(x)=A(x)+A^\dagger(x),\quad \delta(M(x)+iN(x))=-2F(x),\quad \delta\lambda_\alpha(x)=0$$

$$\delta\chi_\alpha(x)=-i\sqrt{2}\psi_\alpha(x),\quad \delta v_\mu(x)=-i\partial_\mu(A(x)-A^\dagger(x)),\quad \delta D(x)=0$$

One can choose the parameters of gauge transformations to set the fields $C(x), M(x), N(X)$ and $\chi_\alpha(x)$ equal to zero. Then one is left with a remaining gauge freedom with the parameter $A(x)=\frac{i}{2}a(x)$. Therefore, we have

$$\delta v_\mu(x)=\partial_\mu a(x),\quad \delta\lambda_\alpha(x)=\delta D(x)=0$$

This gauge is called Wess-Zumino gauge. The vector superfield in this gauge has the form

$$V(x,\theta,\bar\theta)=-\theta^\alpha\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}v(x)_\mu+i(\theta^\alpha\theta_\alpha)\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x)-i(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})\theta^\alpha\lambda_\alpha(x)+\frac{1}{2}(\theta^\alpha\theta_\alpha)(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})D(x) \tag{4}$$

Notice, that in the Wess-Zumino gauge

$$V^2(x,\theta,\bar\theta)=-\frac{1}{2}(\theta^\alpha\theta_\alpha)(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})v^\mu(x)v_\mu(x),\quad V^3(x,\theta,\bar\theta)=0 \tag{5}$$

This supefield is a supersymmetric generalization of a vector potential. From $V(x,\theta,\bar\theta)$ one can build a chiral and an antichiral superfield as

$$\mathcal{W}_\alpha(x,\theta,\bar\theta) = -\frac{1}{4}\bar D_{\dot\alpha}\bar D^{\dot\alpha} D_\alpha\, V(x,\theta,\bar\theta), \quad \overline{\mathcal{W}}_{\dot\alpha}(x,\theta,\bar\theta) = -\frac{1}{4} D^\alpha D_\alpha \bar D_{\dot\alpha}\, V(x,\theta,\bar\theta) \tag{6}$$

The chirality (antichirality) of these superfields follows from the identities $\bar D_{\dot\beta}\bar D_{\dot\alpha}\bar D^{\dot\alpha} = D_\beta D^\alpha D_\alpha = 0$. One can also check, by simply using the algebra of the operators and the (anti)chirality contraints, that (6) is invariant under the gauge transformations given by (3).

An explicit form of $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot\alpha}$ can be obtained by direct calculations

$$\mathcal{W}_\alpha = -i\lambda_\alpha(y) + [\delta_\alpha^\beta D(y) - \frac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta(\partial_\mu v_\nu(y) - \partial_\mu v_\nu(y))]\theta_\beta + \theta^\beta\theta_\beta\, \sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{\dot\beta}(y)$$

$$\overline{\mathcal{W}}_{\dot\alpha} = i\bar\lambda_{\dot\alpha}(y^+) + [\epsilon_{\dot\alpha\dot\beta}D(y^+) - \frac{i}{2}\epsilon_{\dot\alpha\dot\gamma}(\bar\sigma^\mu\sigma^\nu)^{\dot\gamma}{}_{\dot\beta}(\partial_\mu v_\nu(y^+) - \partial_\mu v_\nu(y^+))]\bar\theta^{\dot\beta} - \epsilon_{\dot\alpha\dot\beta}\bar\theta_{\dot\gamma}\bar\theta^{\dot\gamma}\,(\bar\sigma^\mu)^{\dot\beta\alpha}\partial_\mu\lambda_{\dot\alpha}(y^+)$$

where again $y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta_{\dot\alpha}$ and $y^{+,\mu} = x^\mu - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta_{\dot\alpha}$

We can build SUSY invariant Lagrangians similarly to how we have done for chiral superfields,

$$L = \int d^2\theta\, \mathcal{W}^\alpha\mathcal{W}_\alpha + \int d^2\bar\theta\, \overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha} \tag{7}$$

After integration over the Grassmann variables one gets

$$L = \frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha}$$

Finally let us write the SUSY trasformations fo the component fields

$$\begin{aligned}
\delta F_{\mu\nu}(x) &= i\epsilon^\alpha(\sigma_\nu)_{\alpha\dot\alpha}\,\partial_\mu\bar\lambda^{\dot\alpha}(x) + i\bar\epsilon^{\dot\alpha}(\sigma_\nu)_{\alpha\dot\alpha}\,\partial_\mu\lambda^\alpha(x) - (\mu\leftrightarrow\nu)\\
\delta\lambda_\alpha(x) &= i\epsilon_\alpha\, D(x) + (\sigma^{\mu\mu})_\alpha{}^\beta\,\epsilon_\beta\, F_{\mu\nu}(x)\\
\delta D(x) &= \bar\epsilon^{\dot\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu\lambda^\alpha(x) - \epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\,\partial_\mu\bar\lambda^{\dot\alpha}(x)
\end{aligned} \tag{8}$$

The field $D(x)$ is auxiliary. It does not have a kinetic term and transforms as a total derivative under SUSY transformations. Therefore, one can add a term ξD to the Lagrangian, where ξ is a constant. This term is called Fayet-Iliopoulos term.

Let us note, that it is not possible to choose the Wess-Zumino gauge for a massive vector superfield. In case of massive vector superfield one has to add a term

$$m^2\int d^2\theta\, d^2\bar\theta\; V^2$$

to the Lagrangian (7). The absence of the Wess-Zumino gauge for the massive vector superfield can be easily understood, since we need extra degrees of freedom comparing to the massless case. these degrees of freedom are provided by the fields $C(x)$ and $\chi(x)$.

2. Gauge Invariant Theory

2.1. **Gauge transformations.** Chiral superfields Φ_l transform under global $U(1)$ transformations as

$$\Phi_l' = e^{-it_l\Lambda}\Phi_l \tag{9}$$

where t_l are corresponding $U(1)$ charges. In the previous lecture we considered a SUSY invariant Lagrangian for chiral superfields. In order the lagrangian to be invariant under (9) one requires that $m_{ij} = 0$ and $\lambda_{ijk} = 0$ whenever $t_i + t_j \neq 0$ and $t_i + t_j + t_k \neq 0$. Also $k_i = 0$.

In order to make global transformations (9) to be local ones, we take the parameter Λ to be a chiral superfield. In this way one preserves the chirality of Φ_l'. Similarly to as it happens in nonsupersymmetric field theories, when we pass from global to local transfromations, we have to restore invariance of the kinetic term, whereas potential terms stay invariant.

Kinetic terms for chiral superfields transform as

$$\Phi_l^\dagger \Phi_l \to \Phi_l^\dagger \Phi_l e^{it_l(\Lambda^+ - \Lambda)}$$

For the vector superfield we have transformations (3). Therefore, the expression

$$\int d^2\theta d^2\bar{\theta}\Phi_l^\dagger e^{t_l V}\Phi_l \tag{10}$$

is gauge invariant.

Recall, that the terms with superpotential for the chiral superfields are

$$\int d^2\theta W(\Phi_i) + \int d^2\bar{\theta}\bar{W}(\Phi_i^\dagger) \tag{11}$$

The kinetic terms for the gauge field have the form (7). The Lagrangian, which is the sum of (10), (11) and (7) describes an interaction of a supersymmetric matter (described by (anti)chiral superfields) with an $U(1)$ gauge field $v_\mu(x)$ and with its superpartner $\lambda_\alpha(x)$. The later is sometimes called gaugino.

2.2. **Supersymmetric Electrodynamics.** To build supersymmetric electrodynamics we need two chiral superfields Φ_+ and Φ_- which transform as

$$\Phi_+' = e^{-ie\Lambda}\Phi_+, \quad \Phi_-' = e^{ie\Lambda}\Phi_-$$

The Lagrangian is

$$\begin{aligned} L \;=\; & \int d^2\theta\, \mathcal{W}^\alpha \mathcal{W}_\alpha + \int d^2\bar{\theta}\, \overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha} + \\ + \; & \int d^2\theta d^2\bar{\theta}\Phi_+^\dagger e^{eV}\Phi_+ + \int d^2\theta d^2\bar{\theta}\Phi_-^\dagger e^{-eV}\Phi_- + \\ + \; & m\int d\theta^2 \Phi_+\Phi_- + m\int d\bar{\theta}^2 \Phi_+^\dagger \Phi_-^\dagger \end{aligned} \tag{12}$$

Notice, we had to introduce two chiral superfields, in order to write a mass term. In a nonSUSY field theory we could use the complex conjugation $m^2\phi^\star\phi$. In SUSY field theories we can not do so, since a superpotential must be holomporphic i.e., contain only chiral superfields.

One can rewrite the Lagrangian (12) in terms of the component fields, and exclude auxiliary fields $F_+(x), F_-(x)$ and $D(x)$ via their own equations of motion. It is possible to see from the Lagrangian written in terms of the component fields, that the Weyl fermions ψ_+ and ψ_- will form a massive Dirac fermion.

2.3. **Nonabelian Gauge Symmetry.** In this case the superfields Φ, Λ and V are matrix valued. The generalization of (6) is given by

$$\mathcal{W}_\alpha(x,\theta,\bar\theta) = -\frac{1}{4}\bar D_{\dot\alpha}\bar D^{\dot\alpha} e^{-V(x,\theta,\bar\theta)} D_\alpha\, e^{V(x,\theta,\bar\theta)}$$

and it transforms under gauge transformations as

$$\mathcal{W}_\alpha \to e^{-i\Lambda}\mathcal{W}_\alpha e^{i\Lambda}$$

One can use the Wess -Zumino gauge for nonabelian vector superfields as well. Gauge transformations for the vector superfield are

$$e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}$$

and the gauge invariant Lagrangian is again the sum of (10), (11) and (7).