LECTURE 11. CHIRAL SUPERFIELDS

1. Superspace

As we discussed in the previous lecture, SUSY algebra is an extension of the Poincare algebra. In Poincare algebra the four momentum P_μ corresponds to translations in space time: $x^\mu \to x^\mu + a^\mu$. SUSY generators Q_α and $\bar{Q}_{\dot\alpha}$ correspond to translations in an extended space, which is called superspace.

Superspace contains x^μ and coordinates θ_α and $\bar\theta_{\dot\alpha}$. The later are Grassmann variables i.e, they satisfy anticommutation relations

$$\{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = 0$$

from these equations one can conclude that $(\theta_1)^2 = (\theta_2)^2 = (\bar\theta_{\dot 1})^2 = (\bar\theta_{\dot 2})^2 = 0$, i.e., the Grassmann variables are nilpotent.

A group element is[1]

$$G(x,\theta,\bar\theta) = e^{i(-x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar\theta^{\dot\alpha}\bar{Q}_{\dot\alpha})}$$

Multiplying two group elements

$$G(0,\xi,\bar\xi)\, G(x,\theta,\bar\theta) = G(x + i\theta\sigma\bar\xi - i\xi\sigma\bar\theta, \theta+\xi, \bar\theta+\bar\xi)$$

Therefore we have the transformation rule

$$x^\mu \to x^\mu + i\theta^\alpha\, \sigma^\mu_{\alpha\dot\alpha}\, \bar\xi^{\dot\alpha} - i\xi^\alpha\, \sigma^\mu_{\alpha\dot\alpha}\, \bar\theta^{\dot\alpha}$$

$$\theta_\alpha \to \theta_\alpha + \xi_\alpha, \quad \bar\theta_{\dot\alpha} \to \bar\theta_{\dot\alpha} + \bar\xi_{\dot\alpha}$$

The corresponding generators have the form[2]

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}, \quad \bar{Q}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha}\frac{\partial}{\partial x^\mu} \tag{1}$$

One can check, that

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

Let us introduce supercovariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu}, \quad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu} \tag{2}$$

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[1] Complex conjugation rule is $(\Theta_1\Theta_2)^\star = \Theta_2^\star\Theta_1^\star$.

[2] We are considering the left differentiation, i.e., the differentiation operators act from the left. It is important to remember, that for Grassmann variables Θ_i we have $\{\frac{\partial}{\partial\Theta_i}, \Theta_j\} = \delta_{ij}$.

Supercovariant derivatives anticommute with the SUSY generators

$$\{Q_\alpha, D_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = \{Q_\alpha, \bar{D}_{\dot{\beta}}\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0$$

and satisfy the algebra

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial x^\mu}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0$$

2. Superfields

Since we are considering a superspace, it is natural to consider superfields, which are defined on a superspace. A superfield has a form $\mathcal{F}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. A dependence on the Grassmann variables is understood as follows. The superfield is expanded in terms of Grassmann variables as

$$\mathcal{F}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \varphi(x^\mu) + \theta^\alpha \psi_\alpha(x^\mu) + \bar{\theta}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}(x^\mu) + \theta^\alpha \theta^\beta \tau_{\alpha\beta}(x^\mu) + ... \tag{3}$$

where $\varphi, \psi_\alpha, \bar{\chi}_{\dot{\alpha}}, \tau_{\alpha\beta}$ etc., depend only on x^μ. These fields are called component fields. Since Grassman variables are nilpotent the expansion (3) will terminate i.e., we have a finite number of component fields.

A superfield $\mathcal{F}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ can have a number of spinorial or vectorial indices, as well as internal indices. The same indices will appear for the component fields as well. If a superfield is bosonic, then the component fields which are multiplied by even powers of θ- variables are bosonic, and the other ones are fermionic.

A supefield transforms under supersymmetry transformations as

$$\delta\mathcal{F}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = (\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) \mathcal{F}(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \tag{4}$$

where ϵ_α and $\bar{\epsilon}_{\dot{\alpha}}$ are constant parameters of SUSY transformations.

3. Chiral Superfield

One can reduce the number of component fields imposing some additional constraints on a superfield.

A chiral superfield obeys the constraint

$$\bar{D}_{\dot{\alpha}} \Phi(x, \theta, \bar{\theta}) = 0 \tag{5}$$

This constraint is invariant under SUSY transformations, because the supercovariant derivative anticommutes with SUSY generators.

One can solve (5) by noticing

$$\bar{D}_{\dot{\alpha}}(x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}) = 0, \quad \bar{D}_{\dot{\alpha}} \theta^\alpha = 0$$

Therefore a chiral superfield has a form $\Phi(y^\mu, \theta_\alpha)$, where $y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$

We have

$$\Phi(y^\mu,\theta_\alpha) = A(y)+\sqrt{2}\theta^\alpha\psi_\alpha(y^\mu)+\theta^\alpha\theta_\alpha F(y) = \tag{6}$$
$$= A(x)+i\theta^\alpha\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\partial_\mu A(x)+\frac{1}{4}(\theta^\alpha\theta_\alpha)(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})\Box A(x)+$$
$$+ \sqrt{2}\theta^\alpha\psi_\alpha(x)-\frac{i}{\sqrt{2}}(\theta^\alpha\theta_\alpha)\partial_\mu\psi^\beta(x)\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}+\theta^\alpha\theta_\alpha F(x)$$

Antichiral superfields obey the constraint

$$D_\alpha\Phi^\star(x,\theta,\bar\theta)=0 \tag{7}$$

This equation can be solved in a similar way as (5). Namely the antichiral superfield has the form $\Phi^\star(y^{+,\mu},\bar\theta_{\dot\alpha})$, where $y^{+,\mu}=x^\mu-i\theta^\alpha\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}$. Therefore,

$$\Phi^\star(y^{+,\mu},\bar\theta^{\dot\alpha}) = A^\star(y^+)+\sqrt{2}\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(y^+)+\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}F^\star(y^+) = \tag{8}$$
$$= A^\star(x)-i\theta^\alpha\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\partial_\mu A^\star(x)+\frac{1}{4}(\theta^\alpha\theta_\alpha)(\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha})\Box A^\star(x)+$$
$$+ \sqrt{2}\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x)+\frac{i}{\sqrt{2}}(\bar\theta_{\dot\alpha}\bar\theta_{\dot\alpha})\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\,\partial_\mu\bar\psi^{\dot\beta}(x)+\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}F^\star(x)$$

Since the constraints (5) and (7) are complex conjugate to each other, the component fields of the chiral (6) and antichiral (8) superfields are complex conjugate to each other.

A product of two chiral superfields is also chiral. For example if Φ_1 is bosonic, then

$$\bar D_{\dot\alpha}(\Phi_1\Phi_2)=\bar D_{\dot\alpha}\Phi_1\,\Phi_2+\Phi_1\bar D_{\dot\alpha}\Phi_2=0$$

Let us consider SUSY transformations of a chiral superfield. From (4) we have

$$\delta\Phi(x^\mu,\theta_\alpha,\bar\theta_{\dot\alpha})=(\epsilon^\alpha Q_\alpha+\bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\Phi(x^\mu,\theta_\alpha,\bar\theta_{\dot\alpha}) \tag{9}$$

Using the explicit form of the supercharges (1) and of the superfield[3] (6) and equating the same powers of the θ - variables we get

$$\begin{aligned}\delta A(x)&=\sqrt{2}\,\epsilon^\alpha\,\psi_\alpha(x)\\ \delta\psi_\alpha(x)&=i\sqrt{2}\,\sigma^\mu_{\alpha\dot\alpha}\,\bar\epsilon^{\dot\alpha}\,\partial_\mu A(x)+\sqrt{2}\,\epsilon_\alpha\,F(x)\\ \delta F(x)&=i\sqrt{2}\,\bar\epsilon_{\dot\alpha}\,(\bar\sigma^\mu)^{\alpha\dot\alpha}\,\partial_\mu\psi_\alpha(x)\end{aligned} \tag{10}$$

Let us notice, that the component field $F(x)$ transforms as a total derivative.

It is easier to work in the y basis. The supersymmetry generators and covariant derivatives in this baisis are

$$Q_\alpha=\frac{\partial}{\partial\theta^\alpha},\quad \bar Q_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta_{\dot\alpha}}+2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} \tag{11}$$

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu},\quad \bar D_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta_{\dot\alpha}} \tag{12}$$

[3]In the LHS of (9) we write $\delta\Phi=\delta A+\sqrt{2}\theta^\alpha\delta\psi_\alpha+\theta^\alpha\theta_\alpha\delta F+\ldots$

Now, let us construct an action, which is invariant under the transformations (10). To this end we need an integration measure in the superspace. Let us recall that integration with respect to Grassmann variables is the same as differentiation

$$\int d\Theta\, \Theta = 1, \quad \int d\Theta = 0$$

Let us define

$$d^2\theta = -\frac{1}{4} d\theta^\alpha\, d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\bar{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}}\, d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}},$$

Then

$$\int d^2\theta\, \theta^2 = 1, \quad \int d^2\bar{\theta}\, \bar{\theta}^2 = 1, \quad \theta^2 \equiv \theta^\alpha \theta_\alpha, \quad \bar{\theta}^2 \equiv \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$$

The expression

$$\int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\Phi^\star) \tag{13}$$

is invariant under SUSY transformations (9). Here $W(\Phi)$ is an arbitrary function of chiral superfields. This function is called a superpotential. Similarly, $\bar{W}(\Phi^\star)$ depends only on antichiral superfields.

First, let us notice that the integration measure in the superspace is invariant under SUSY transformations, since it is a translation in the superspace. Let us consider the variation of the first term in (13) under the SUSY transformations with the parameter ϵ_α

$$\delta\left(\int d^2\theta W(\Phi)\right) = \int d^2\theta \frac{\partial W(\Phi)}{\partial \Phi} \delta\Phi = \int d^2\theta \frac{\partial W(\Phi)}{\partial \Phi} \epsilon^\alpha \frac{\partial}{\partial \theta^\alpha} \Phi = \int d^2\theta \frac{\partial}{\partial \theta^\alpha} W(\Phi) = 0$$

because we get the differentiation with respect to three θ-s.

Similarly one can prove the invarinace with respect to variation with the parameter $\bar{\epsilon}_{\dot{\alpha}}$. Using $\bar{Q}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} + 2i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \frac{\partial}{\partial y^\mu}$ and the chirality constraint (5) we get

$$\delta\left(\int d^2\theta W(\Phi)\right) = \int d^2\theta \frac{\partial W(\Phi)}{\partial \Phi} \bar{\epsilon}_{\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \left(2i\sigma^\mu_{\alpha\dot{\beta}} \theta^\alpha \frac{\partial}{\partial y^\mu}\right) \Phi = -2i\sigma^\mu_{\alpha\dot{\beta}} \int d^2\theta \bar{\epsilon}^{\dot{\alpha}} \theta^\alpha \partial_\mu W(\Phi)$$

and is a total derivative. One can also prove that the expression

$$\int d^2\theta\, d^2\bar{\theta}\ K(\Phi, \Phi^\star) \tag{14}$$

where $K(\Phi, \Phi^\star)$ is a real function of chiral and antichiral superfields is invariant under SUSY transformations. This function is called Kähler potential.

The total Lagrangian has therefore the form

$$L = \int d^2\theta\, d^2\bar{\theta}\ K(\Phi, \Phi^\star) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\Phi^\star) \tag{15}$$

It does not change if we write it in x- or y- basis.

The forms of a superpotential and of the Kähler potential can be restricted by the requirement of the renormalizability, i.e. the dimensions of the parameters (masses, coupling constants) should be non-negative. From SUSY algebra it is clear, that θ has the dimension

$-\frac{1}{2}$. Therefore, $\int d^2\theta$ and $\int d^2\bar{\theta}$ (it is a differentiation) have the dimension $\frac{1}{2}$. Similarly, $\int d^2\theta d^2\bar{\theta}$ has the dimension 1.

When a chiral superfield has a dimension 1, then the Kähler potential and the superpotential are taken to have the form

$$K = g\Phi\,\Phi^\star$$
$$W = \lambda\Phi^3 + m\Phi^2 + k\Phi$$

where λ, m, k and g are constants. If we have several superfields Φ^i, then[4]

$$K = \Phi^i\,\Phi^{i\star}$$
$$W = \lambda_{ijk}\Phi^i\Phi^j\Phi^k + m_{ij}\Phi^i\Phi^i + k_i\Phi^i$$

In order to obtain the expression of the Lagrangian in terms of the component fields, one has to insert the explicit form of the chiral and antichiral superfields into (15) and perform the Grassmann integration

$$\begin{aligned} L &= A_i^\star\Box A_i + i\partial_\mu\bar{\psi}_{i,\dot{\alpha}}(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}}\psi_{i,\alpha} + F_i^\star F_i + \\ &+ (m_{ij}(A_iF_j - \frac{1}{2}\psi_i^\alpha\psi_{j,\alpha}) + \lambda_{ijk}(A_iA_jF_k - \psi_i^\alpha\psi_{j,\alpha}A_k) + k_iF_i + h.c.) \end{aligned} \tag{16}$$

As we can see the component field F_i does not have a kinetic term. This field is auxiliary. It can be expressed via its own equation of motion

$$\frac{\partial L}{\partial F_l^\star} = F_l + k_l + m_{il}A_i^\star + \lambda_{ijl}A_i^\star A_j^\star = 0$$

and put back into the Lagrangian (16)

$$\begin{aligned} L &= A_i^\star\Box A_i + i\partial_\mu\bar{\psi}_{i,\dot{\alpha}}(\bar{\sigma}^\mu)^{\alpha\dot{\alpha}}\psi_{i,\alpha} \\ &- \frac{1}{2}m_{ij}(\psi_i^\alpha\psi_{j,\alpha} + \bar{\psi}_{i,\dot{\alpha}}\bar{\psi}_i^{\dot{\alpha}}) - \lambda_{ijk}(\psi_i^\alpha\psi_{j,\alpha}A_k + \bar{\psi}_{i,\dot{\alpha}}\bar{\psi}_i^{\dot{\alpha}}A_k^\star) - V(A_i, A_j^\star) \end{aligned} \tag{17}$$

where

$$V(A_i, A_j^\star) = F_k^\star F_k$$

The purpose of the auxiliary field F is that it ensures the closure of the SUSY algebra without a use of equations of motion (off-shell closure). Otherwise, without the auxiliary field the SUSY algebra will close only if we use the equations of motion for fermions (on-shell closure). With the auxiliary field we have equal number of bosonic and fermionic degrees of freedom matches also off-shell. Indeed, a complex scalar has two degrees of freedom, the Weyl fermion has four degrees of freedom and a complex auxiliary field provides extra two bosonic degrees of freedom.

[4] We took the Kähler potential to be diagonal in terms of the fields