LECTURE 10. SUSY ALGEBRA IN $D = 4$

1. Two component Weyl spinors

Let us consider the Poincare algebra in $D = 4$. It consists of translations

$$P_\mu = -i\partial_\mu,$$

and of four dimensional Lorentz rotations

$$L_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

where $\mu = 0, 1, 2, 3$. We have total 10 generators,

(1) $$[P_\mu, P_\nu] = 0, \quad [L_{\mu\nu}, P_\rho] = i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu$$

(2) $$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\mu\rho}L_{\sigma\nu} + i\eta_{\mu\sigma}L_{\nu\rho} + i\eta_{\nu\rho}L_{\mu\sigma} + i\eta_{\nu\sigma}L_{\mu\rho}$$

Let us consider an extension of this algebra with extra generators. Recall, that the four dimensional Lorentz group, which keeps the interval $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ invariant, is $SO(3,1)$. The later is locally isomporphic to the group $SL(2,C)$. This group consist of the complex 2×2 matrices with unit determinant

(3) $$M_\alpha{}^\beta = \begin{pmatrix} Z_1{}^1 & Z_1{}^2 \\ Z_2{}^1 & Z_2{}^2 \end{pmatrix}, \quad Z_1{}^1 Z_2{}^2 - Z_1{}^2 Z_2{}^1 = 1$$

Apparently, this group has 6 independent parameters, as the four dimensional Lorentz group should have.

The fundamental representation of the $SL(2,C)$ group is a complex two dimensional spinor ψ_α, with $\alpha = 1, 2$. It transforms as

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta, \quad \bar\psi'_{\dot\alpha} = (M^\star)_{\dot\alpha}{}^{\dot\beta}\bar\psi_{\dot\beta}$$

where $(M^\star)_{\dot\alpha}{}^{\dot\beta}$ is a complex conjugate to $M_\alpha{}^\beta$ and the spinor $\bar\psi_{\dot\alpha}$ (with $\dot\alpha = 1, 2$) is a complex conjugate to the spinor ψ_α. These spinorial representations (ψ_α and $\bar\psi_{\dot\alpha}$) are often denoted as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

Let us consider also 2×2 matrices

(4) $$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and also the matrices $\epsilon_{\dot\alpha\dot\beta}$ which have the same form as undotted ones. These matrices are called spinor metric. Using spinor metric one can lift and lower the spinor indices.

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\psi_{\dot\beta}, \quad \psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}$$

Using (4), one has $\epsilon_{\alpha\beta}\,\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$ and $\epsilon_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}$.

The spinor metric is invariant under $SL(2,C)$ group transformations

$$\epsilon'_{\alpha\beta} = M_\alpha{}^\gamma M_\alpha{}^\beta \epsilon_{\gamma\delta}$$

The invariance can be easily proven, using (3) and (4).

Spinors ψ^α and $\psi^{\dot\alpha}$ transform under the $SL(2,C)$ group transformations as

$$(\psi^\alpha)' = (M^{-1})_\beta{}^\alpha \psi^\beta, \quad (\bar\psi^{\dot\alpha})' = (M^{-1})^\star{}_{\dot\beta}{}^{\dot\alpha}\bar\psi^{\dot\beta}$$

Therefore $\psi^\alpha\,\psi_\alpha$ and $\bar\psi^{\dot\alpha}\,\bar\psi_{\dot\alpha}$ are Lorentz invariant.

Multiplication of the representations goes the same way as in Quantum Mechanics, when we find a spin of composite systems. For example

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right) = (1,0)\oplus(0,0)$$

which is $\psi_\alpha\psi_\beta \sim \psi_{(\alpha\beta)} + \psi_{[\alpha\beta]}$, a sum of symmetric and anysymmetric representations.

Similarly

$$(1,0)\otimes\left(\frac{1}{2},0\right) = \left(\frac{3}{2},0\right)\oplus\left(\frac{1}{2},0\right)$$

Also

$$\left(\frac{1}{2},0\right)\otimes\left(0,\frac{1}{2}\right) = \left(\frac{1}{2},\frac{1}{2}\right)$$

The representation $\left(\frac{1}{2},\frac{1}{2}\right)$ describes a four- vector: $\psi_\alpha\psi_{\dot\alpha} \sim \psi_{\alpha\dot\alpha}$

A four- vector $\psi_{\alpha\dot\alpha}$ in spinorial notations is connected to a four - vector ψ_μ in tensorial notations as

$$\psi_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}\,\psi_\mu$$

where $\sigma^\mu = (-1,\sigma^i)$ and σ^i are usual Pauli matrices. Any four vector (the energy momentum vector, for example) can therefore be written as[1]

$$P_{\alpha\dot\alpha} = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix}$$

Dirac spinor in these notations is written as

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi_{\dot\alpha} \end{pmatrix}$$

Majorana spinor is

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\psi_{\dot\alpha} \end{pmatrix}$$

[1]The metric is mostly plus $\eta_{\mu\nu} = (-1,1,1,1)$

2. Super Poincare Algebra

Let us extend the Poincare algebra with generators ${Q_\alpha}^A$ and $\bar{Q}_{\dot\alpha B}$. Here the indices α and $\dot\alpha$ are spinorial indices, considered above, while A and B are some extra indices $A, B = 1, 2, ..., N$.

The generators ${Q_\alpha}^A$ and $\bar{Q}_{\dot\alpha B}$ are anticommuting. They satisfy

$$\{{Q_\alpha}^A,\, \bar{Q}_{\dot\alpha B}\} = 2\,(\sigma^\mu)_{\alpha\dot\alpha}\, P_\mu\, \delta^A_B \tag{5}$$

$$[P_\mu,\, {Q_\alpha}^A] = [P_\mu,\, \bar{Q}_{\dot\alpha A}] = 0 \tag{6}$$

This algebra is an example of so-called graded Lie algebras. These algebras contain even $\mathcal{E}$ and odd $\mathcal{O}$ generators and have a structure

$$[\mathcal{E},\, \mathcal{E}]_- = \mathcal{E}, \quad [\mathcal{O},\, \mathcal{O}]_+ = \mathcal{E}, \quad [\mathcal{E},\, \mathcal{O}]_- = \mathcal{O}$$

In addition to (5) and (6) Super Poincare algebra (we shall refer to it simply as SUSY algebra) in general contains the antocommutators

$$\{{Q_\alpha}^A,\, {Q_\beta}^B\} = \epsilon_{\alpha\beta}\, Z^{AB}, \quad \{\bar{Q}_{\dot\alpha A},\, \bar{Q}_{\dot\beta B}\} = \epsilon_{\dot\alpha\dot\beta} Z_{AB} \tag{7}$$

The generators Z^{AB} and Z_{AB} are central charges of the algebra. Their commutators with all other elements of the algebra and and among themselves is zero.

Let us consider the first equation in (7). The L.H.S is symmetrical with respect to the simultaneous interchange of indexes (α, A) and (β, B). The R.H.S is antisymmetric with respect to α and β. Therefore it should be antisymmetric with respect to A and B. Therefore the central charges Z^{AB} can be nonzero only if $N > 1$. SUSY algebras with $N > 1$ are called extended algebras. Extended SUSY algebras can be either with or without central charges.

The commutation relations between Lorentz rotations and the supercharges are given by

$$[L_{\mu\nu},\, {Q_\alpha}^A] = -{(\sigma_{\mu\nu})_\alpha}^\beta {Q_\beta}^A, \quad [L_{\mu\nu},\, \bar{Q}_{\dot\alpha A}] = -{(\bar\sigma_{\mu\nu})_{\dot\alpha}}^{\dot\beta} \bar{Q}_{\dot\beta A} \tag{8}$$

where

$${(\sigma_{\mu\nu})_\alpha}^\beta = \frac{1}{4}((\sigma_\mu)_{\alpha\dot\alpha}\, (\bar\sigma_\nu)^{\dot\alpha\beta} - (\sigma_\nu)_{\alpha\dot\alpha}\, (\bar\sigma_\mu)^{\dot\alpha\beta})$$

$${(\bar\sigma_{\mu\nu})^{\dot\alpha}}_{\dot\beta} = \frac{1}{4}((\bar\sigma_\mu)^{\dot\alpha\alpha}\, (\sigma_\nu)_{\alpha\dot\beta} - (\bar\sigma_\nu)^{\dot\alpha\alpha}\, (\sigma_\mu)_{\alpha\dot\beta})$$

and[2]

$$(\bar\sigma_\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\, \epsilon^{\alpha\beta}\, (\sigma_\mu)_{\beta\dot\beta}$$

Finally, SUSY algebra contains generators B_l. These generators correspond to the internal-global symmetries of the algebra. They are often called R- symmetry. These generators are Lorentz scalars and satisfy

$$[\mathcal{B}_l,\, \mathcal{B}_k] = i C^j_{lk} \mathcal{B}_j \tag{9}$$

[2]The sigma matrices satisfy $Tr(\sigma^\mu \bar\sigma^\nu) = -2\eta^{mn}$ and $(\sigma_\mu)_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\beta\beta} - 2\delta^\beta_\alpha \delta^{\dot\beta}_{\dot\alpha}$

where C_{lk}^{j} are structure constants. One has

$$[\mathcal{B}_l,\, Z_{AB}] = [\mathcal{B}_l,\, Z^{AB}] = 0 \tag{10}$$

$$[Q_\alpha{}^A,\, \mathcal{B}_l] = (S_l)^A{}_B Q_\alpha{}^B, \quad [\bar{Q}_{\dot{\alpha}A},\, \mathcal{B}_l] = (S_l^\star)_A{}^B \bar{Q}_{\dot{\alpha}B} \tag{11}$$

Here $(S_l)^A{}_B$ and $(S_l^\star)_A{}^B$ are matrix representations of the generators $\mathcal{B}_l$.

To summarize: the equations (1)–(2) and (5)– (11) are the most general super Poincare algebra.

3. Representations of $N = 1$ SUSY algebra

3.1. **Casimir Operators.** As for any algebra, here we can also build the representations. They can be either reducible or irreducible.

Irreducible representations are characterized by eigenvalues of the Casimir generators. Recall, that for the $D = 4$ Poincare algebra has two Casimir generators

- $P^2 = P^\mu P_\mu$. It is a mass square operator whose eigenvalues are $-m^2$
- $W^\mu W_\mu$, where W_μ is the Pauli - Lubanski vector.

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu L^{\rho\sigma}$$

  The eigenvalues of $W^\mu W_\mu$ are $m^2 s(s+1)$ with $s = 0, \frac{1}{2}, 1...$ for massive representations and $W_\mu = \lambda P_\mu$ for massless representations.

In case of the Super Poincare algebra, we again have two Casimir generators. One is again P^2, since it apparently commutes with all generators of the algebra.

The other one has the form

$$C^2 = C^{\mu\nu} C_{\mu\nu} \tag{12}$$

where

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu, \quad B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} Q_\alpha$$

The fact, that (12) commutes with Q_α can be checked using

$$\epsilon^{\mu\nu\rho\tau}(\sigma_{\rho\tau})_\alpha{}^\beta = 2i(\sigma^{\mu\nu})_\alpha{}^\beta, \quad \epsilon^{\mu\nu\rho\tau}(\bar{\sigma}_{\rho\tau})^{\dot{\alpha}}{}_{\dot{\beta}} = -2i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}},$$

3.2. **Massive Representations.** Let us move to the rest frame where $P_\mu = (-m, 0, 0, 0)$. In this reference frame one gets

$$C^2 = 2m^4 J^i J_i, \quad J_i = S_i - \frac{1}{4m}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_i)^{\dot{\alpha}\alpha} Q_\alpha, \quad i = 1, 2, 3 \tag{13}$$

We have $[J_i,\, J_j] = i\epsilon_{ijk} J_k$, since the operators S_i and the Pauli matrices $\bar{\sigma}_i$ satisfy the same relations. We have

$$\{Q_\alpha,\, \bar{Q}_{\dot{\alpha}}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

Let us take a state $|m, j\rangle$, which is characterized by the quantum numbers m and j. One can generate a new state $|\Omega\rangle$ as

$$|\Omega\rangle = Q_1\, Q_2 |m, j\rangle$$

Obviously

$$Q_1|\Omega\rangle = Q_2|\Omega\rangle = 0$$

This state is called Clifford vacuum. Indeed if we consider the operators $\bar{Q}_{\dot{1}}, \bar{Q}_{\dot{2}}$ as creators and the operators Q_1, Q_2 as annihilators, then it will look like an ordinary vacuum. The difference is that the Clifford vacuum can have nonzero quantum numbers. Notice from (13) that the action of J_i on the Clifford vacuum coincides with the one of S_i. Therefore the Clifford vacuum is characterized by a mass and a spin (as it is for a representation of the Poincare group) and describes an asymptotic (physical) state

$$|\Omega\rangle = |m, s, s_3\rangle$$

Let us introduce operators

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2}, \quad a^+_{1,2} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}$$

$$\{a_1, a_1^+\} = \{a_2, a_2^+\} = 1$$

Therefore, one has the following states

$$|\Omega\rangle, \quad a_1^+|\Omega\rangle, \quad a_2^+|\Omega\rangle, \quad a_1^+ a_2^+|\Omega\rangle$$

When the Clifford vacuum has the spin 0, one has two states with spin 0. They are $|\Omega\rangle$ and $a_1^+ a_2^+|\Omega\rangle$ (because of antisymmetry with respect to the indices 1 and 2). And we have two states with spin $\frac{1}{2}$. Therefore in this case we have one massive complex scalar and one massive fermion.

Let us notice that whatever the spin assignment for the Clifford vacuum is, the total number of bosonic and fermionic degrees of freedom is equal to each other. Each of this numbers is $2(2s + 1)$.

3.3. **Massless Representations.** Let us choose the frame $P_\mu = (-E, 0, 0, E)$. In this case SUSY algebra is

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

Therefore, we have only one pair of creation and annihilation operators

$$a = \frac{1}{2\sqrt{E}} Q_1, \quad a^+ = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}}, \quad \{a, a^+\} = 1$$

In this case the Clifford vacuum is characterized by the helicity λ. The operator a^+ increases the helicity by $\frac{1}{2}$ and the operator a decreases the helicity by $\frac{1}{2}$. Therefore, we have the states

$$|\Omega\rangle, \quad a^+|\Omega\rangle$$

with helicities λ and $\lambda + \frac{1}{2}$, respectively. Adding CPT conjugated states we eventually get the spectrum with helicities

$$\lambda, \quad \lambda + \frac{1}{2}, \quad -\lambda - \frac{1}{2}, \quad -\lambda$$

Taking for example $\lambda = \frac{1}{2}$, one gets one massless vector field and one massless fermion, two degrees of freedom each. This is called massless vector supermultiplet.

Taking $\lambda = 0$, one gets one massless complex scalar field and one massless fermion, again two degrees of freedom each.

Notice, that whatever the helicity assignment for the Cliffor vacuum is (i.e., whether λ is integer or half-integer) the number of the bosonic and fermionic degrees of freedom are equal to each other.

The property that the total number of bosonic and fermionic degrees of freedom are equal to each other in a supermultiplet holds for extended SUSY algebras as well.