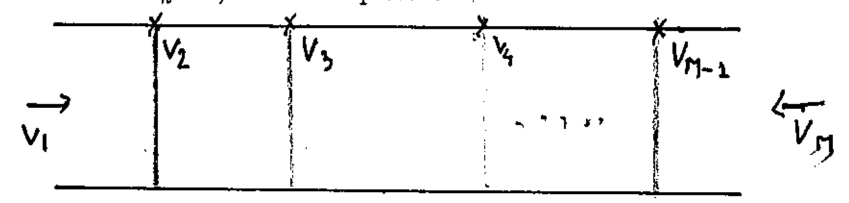
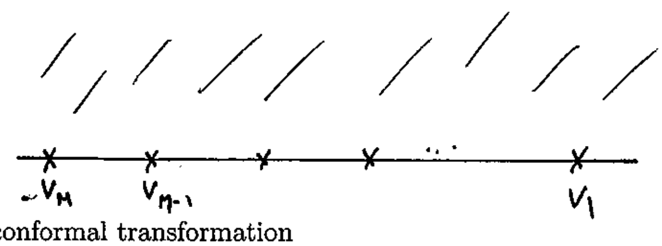
LECTURE 9. STRING PERTURBATION THEORY

1. Open Strings

Let us consider the simplest case, when diagrams do not contain loops, i.e., we have tree level diagrams. Consider Open Strings, M particle scattering amplitude. Let us take the particle number "1" and the particle number "M" and put them in the infinite past and in the infinite future ($\tau \to \pm\infty$) . The other particles are emitted from the boundary of the string $\sigma = 0$ at finite τ_k i.e., we have a picture



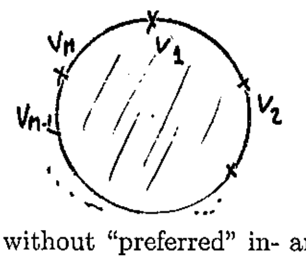
and one integrates over τ- variables. Performing a conformal transformation $v = e^{i(\sigma+i\tau)}$, one mpas the picture above on an upper half-plane.



Next, a conformal transformation

$$\omega = \frac{v-i}{v+i}$$

maps the upper half-plane on a circle



In this way one gets a picture without "preferred" in- and out- states.

In general M point tree-level scattering amplitude has the form

$$A_M = g^{M-2} \langle \phi_1 | V_2(k_2)\, \mathcal{D}\, V_3(k_3) ... \mathcal{D}\, V_{M-1}(k_{M-1}) | | \phi_M \rangle \tag{1}$$

Here $\langle \phi_1 |$ and $|\phi_M\rangle$ are the initial and final asymptotic states. $V_L(k_L)$ are vertex operators, which correspond to the states $2, 3, .., M-1$. They are taken at the point $\tau_L = 0$, since for nonzero τ_L we have

$$V(k,\tau) = e^{i\tau L_0} V(k,0) e^{-i\tau L_0}$$

The propagator for the Open Sting has the form

$$\mathcal{D} = \frac{1}{2(L_0 - 1)}, \quad L_0 = \frac{p^2}{2} + \sum_{n=1}^{\infty} \alpha^{\mu}_{-n} \alpha^{\mu}_{n} \tag{2}$$

We can represent the propagator as

$$\mathcal{D} = \frac{1}{2} \int_0^{\infty} d\tau\, e^{-\tau(L_0 - 1)}$$

The equation (1) is already a complete expression for the given M point amplitudes. There is no need to consider different channels separately, as it is done in the Quantum Field Theory.

Let us consider examples which involve scattering of the open string tachyons and vector bosons. In this case the tachyon vertex operators are given by the eq (5) of the Lecture 8. Let us rewrite them as

$$V_0(k,z) = Z_0 W_0$$

where Z_0 part does not contain oscillators. Recall,that

$$X^{\mu}(z) = x^{\mu} - i p^{\mu} \ln z + i \sum_{n \neq = 0} \frac{\alpha^{\mu}_{n}}{n \cdot} z^{-n}$$

where $z = e^{i\tau}$. Using

$$e^A\, e^B = e^{A+B+\frac{1}{2}[A,B]}$$

which is valid if $[A,[A,B]] = [B,[A,B]] = 0$, we get

$$Z_0 = e^{ik\cdot x}\, z^{k\cdot p+1} = z^{k\cdot p-1}\, e^{ik\cdot x} \tag{3}$$

since $[x^{\mu}, p^{\nu}] = i\eta^{\mu\nu}$.

1.1. Examples.

1.1.1. *Three tachyons.* Using (1) and the explicit form of vertex operators for tachyons, we get

$$A_3 = g\langle 0, k_1 | V_0(k_2) | 0, k_3 \rangle = g \tag{4}$$

because

$$V_0(k_2) \sim e^{ik_2\cdot x}, \quad |0, k_1\rangle \sim e^{ik_1\cdot x}, \quad |0, k_3\rangle \sim e^{ik_3\cdot x}$$

and we omitted the delta-function in (4), which reflects the conservation of the total momentum.

1.1.2. *Two tachyons, one vector boson.* Let us consider tachyons as in - and out- states. Recall, that the vertex operator for the Open String vector boson is

$$V(\xi, k, z) =: \xi \cdot \frac{\partial X(z)}{\partial \tau} e^{ik\cdot X(z)} :$$

where ξ_μ is a transverse polarization vector $k_2 \cdot \xi = 0$ and $(k_2)^2 = 0$. Then using (1) we get

$$A_3 = g\ \xi \cdot k_3$$

1.1.3. *Four tachyons. Veneziano amplitude.* For four tachyon scattering we have

$$A_4 = g^2 \langle 0, k_1 | V_0(k_2)\, \mathcal{D}\, V_0(k_3) | 0, k_4 \rangle \tag{5}$$

Performing the Wick rotation $\tau \to i\tau$ we get

$$V_0(k, 0) = e^{\tau L_0} V_0(k, \tau) e^{-\tau L_0} \tag{6}$$

Inserting (2) and (6) in (5) we get

$$\begin{aligned} A_4 &= g^2 \int_0^\infty d\tau \langle 0, k_1 | V_0(k_2)\, e^\tau e^{-\tau L_0}\, e^{\tau L_0}\, V_0(k_3) e^{-\tau L_0} | 0, k_4 \rangle \\ &= g^2 \int_0^\infty d\tau \langle 0, k_1 | V_0(k_2)\, e^\tau\, V_0(k_3, \tau) e^{-\tau L_0} | 0, k_4 \rangle \end{aligned} \tag{7}$$

Now let us use the fact, that $|0, k_4\rangle$ is a physical state, which means $L_0 |0, k_4\rangle = |0, k_4\rangle$. Therefore, (7) is

$$\begin{aligned} A_4 &= g^2 \int_0^\infty d\tau \langle 0, k_1 | V_0(k_2)\, e^\tau\, V_0(k_3, \tau) e^{-\tau} | 0, k_4 \rangle \\ &= g^2 \int_0^\infty d\tau \langle 0, k_1 | V_0(k_2)\ V_0(k_3, \tau) | 0, k_4 \rangle \end{aligned} \tag{8}$$

Introducing a new variable $z = e^{-\tau}$, we get

$$A_4 = g^2 \int_0^1 \frac{dz}{z} \langle 0, k_1 | V_0(k_2, 1)\ V_0(k_3, z) | 0, k_4 \rangle \tag{9}$$

The part of (9) which does not contain the oscillators is

$$\frac{1}{z} \langle 0 | e^{i(k_1+k_2)\cdot x}\ z^{k_3\cdot(k_3+k_4)-1}\ e^{i(k_3+k_4)\cdot x} | 0 \rangle \tag{10}$$

where we omitted the δ- function $e^{i(k_1+k_2+k_3+k_4)\cdot x}$.

In the integrand we have

$$z^{(k_3^2 + k_4\cdot k_3 - 1)-1} = z^{-\frac{s}{2}-2}, \quad s = -(k_3 + k_4)^2$$

and $k_3^2 = 2$ for the tachyon.

Now let us compute the oscillator part

$$\langle 0 | \exp\left(-k_2 \cdot \sum_{n=1}^\infty \frac{\alpha_{-n}}{n} \right) \times \exp\left(k_3 \sum_{n=1}^\infty \frac{\alpha_n}{n} z^n \right) | 0 \rangle \tag{11}$$

Let us use the expression

$$\langle 0| \left(\sum_{n=1}^{\infty} \frac{\alpha_n^{\mu}}{n} y_1^{-n} \right) \times \left(\sum_{n=1}^{\infty} \frac{\alpha_{-n}^{\nu}}{n} y_2^{n} \right) |0\rangle = \eta^{\mu\nu} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{y_2}{y_1} \right)^2 = -\eta^{\mu\nu} \ln \left(1 - \frac{y_2}{y_1} \right)$$

Therefore, (11) is

$$(1-z)^{k_2 \cdot k_3} = (1-x)^{-\frac{t}{2}-2} \tag{12}$$

Putting together (8), (3) and (12), we get

$$A_4 = g^2 \int_0^1 z^{-\frac{1}{2}s-2} (1-z)^{-\frac{1}{2}t-2} \, dz = g^2 B(-\frac{1}{2}s - 2, -\frac{1}{2}t - 2)$$

Therefore we have reconstructed Veneziano amplitude, which we discussed in the Lecture 1.

2. Closed Strings

Tree level amplitudes for closed strings can be computed in a similar way. In particular, the analog of the equation (1) is now

$$A_M = g^{M-2} \langle \phi_1 | V_2(k_2) \, \mathcal{D} \, V_3(k_3) ... \mathcal{D} \, V_{M-1}(k_{M-1}) || \phi_M \rangle + \quad (permutations) \tag{13}$$

but now we should add explicitly the permutations of the vertex operators. The propagator for the bosonic string is

$$\mathcal{D} = \frac{1}{4\pi} \int_{|z| \leq 1} \frac{dz d\bar{z}}{z \bar{z}} \, z^{(L_0 - 1)} \bar{z}^{(\bar{L}_0 - 1)} \tag{14}$$

The equations analogous to (5) and (9) are

$$A_4 = g^2 \langle 0, k_1 | V_0(k_2) \, \mathcal{D} \, V_0(k_3) | 0, k_4 \rangle + g^2 \langle 0, k_1 | V_0(k_3) \, \mathcal{D} \, V_0(k_2) | 0, k_4 \rangle \tag{15}$$

and

$$\begin{aligned} A_4 \quad &= \quad \frac{g^2}{4\pi} \int_{|z| \leq 1} \frac{dz \, d\bar{z}}{z \bar{z}} \langle 0, k_1 | V_0(k_2, 1, 1) \; V_0(k_3, z, \bar{z}) | 0, k_4 \rangle + \\ &+ \quad \frac{g^2}{4\pi} \int_{|z| \leq 1} \frac{dz \, d\bar{z}}{z \bar{z}} \langle 0, k_1 | V_0(k_3, z, \bar{z}) \; V_0(k_2, 1, 1) | 0, k_4 \rangle \end{aligned} \tag{16}$$

since

$$V(k, z, \bar{z}) = z^{L_0} \bar{z}^{\bar{L}_0} V(k, 1, 1) z^{-L_0} \bar{z}^{-\bar{L}_0}$$