LECTURE 8. STRING PERTURBATION THEORY

1. Vertex operators

As we mentioned a few times, the states in String Theory are described by vertex operators. Let us consider an Open String first. vertex operators are taken at the points $\tau = 0$ or $\tau = \pi$. An operator $A(\tau)$ is said to have a conformal dimension Δ, if under the transformations $\tau \to \tau'(\tau)$ it transforms as

$$A'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^{\Delta} A(\tau)$$

If we have infinitisimal transformations

$$\tau \to \tau' = \tau + \epsilon(\tau)$$

we have

$$\delta A = -\epsilon\, \frac{dA}{d\tau} - \Delta\, A\, \frac{d\epsilon}{d\tau} \tag{1}$$

Now let us take $\epsilon = -ie^{im\tau}$, with m being an integer. The corresponding generators are L_m, (Virasoro generators). We can rewrite (1) as

$$[L_m,\, A(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + m\Delta\right) A(\tau) \tag{2}$$

Expanding $A(\tau)$ as

$$A(\tau) = \sum_{m=-\infty}^{\infty} e^{im\tau} A_m$$

we get from (2)

$$[L_m,\, A_n] = (m(\Delta - 1) - n)\, A_{m+n} \tag{3}$$

Let us notice, that if $\Delta = 1$, we have

$$[L_m,\, A_0] = 0 \tag{4}$$

Suppose, $|\phi\rangle$ is a physical state. This means $(L_m - a\delta_{m,0})|\phi\rangle = 0$ for $m \geq 0$. Then the equation (4) means that $|\phi'\rangle = A_0|\phi\rangle$ is also a physical state. We conclude, that in order to describe a transition from a physical state $|\phi\rangle$ to a physical state $|\phi'\rangle$ by emission of a state described by the operator A_0, the open string vertex operator must have a conformal dimension

$$\Delta = 1$$

Similar consideration for a closed string will give us

$$\Delta = 2$$

Back to open strings. A vertex operator $V(k,\tau)$ describes an emission of a string at time τ and at the point $\sigma = 0$. The momentum of the emitted string is $-k_\mu$ and the momentum of the absorbed string is $+k_\mu$. Therefore the vertex operator changes the momentum of a state from which it is emitted (to which it is absorbed) by k_μ. Therefore the dependence on the center of mass coordinate should be described by the operator

$$e^{ik\cdot x(\tau)}, \quad x^\mu(\tau) = x_0^\mu + p^\mu \tau$$

Therefore vertex operators should contain a factor $e^{ik\cdot X(\tau)}$.

Let us find the conformal dimension of the operator $: e^{ik\cdot X(0,\tau)} :$. The symbol :: means normal ordering. In our case the normal ordered expression for the vertex operator is written as

$$V(k,\tau) =: e^{ik\cdot X(0,\tau)} := \exp\left(k_\mu \sum_{n=1}^{\infty} \frac{\alpha_{-n}^\mu}{n} e^{-in\tau}\right) \times e^{ik_\mu x^\mu} \times \exp\left(-k_\mu \sum_{n=1}^{\infty} \frac{\alpha_n^\mu}{n} e^{in\tau}\right) \tag{5}$$

We would like to compute $[L_m, V(k,\tau)]$. Let us use

$$[\alpha_\rho^\mu,\, e^{k\cdot\alpha_{-n}}] = pk^\mu\, \delta_{p-n,0}\, e^{k\cdot\alpha_{-n}}$$

Further, from the expression $L_m = \frac{1}{2}\sum \alpha_{m-q}\cdot\alpha_q$ we obtain

$$[L_m, e^{k\cdot\alpha_{-n}}] = \frac{1}{2}\sum_q (\alpha_{m-q}^\mu\, [\alpha_q^\mu,\ e^{k\cdot\alpha_{-n}}] + [\alpha_{m-q}^\mu\ ,\ e^{k\cdot\alpha_{-n}}]\alpha_q^\mu) = \frac{1}{2} n (k\cdot\alpha_{m-n}\ e^{k\cdot\alpha_{-n}} + e^{k\cdot\alpha_{-n}} k\cdot\alpha_{m-n},) \tag{6}$$

When computing the commutator using (6) there will appear some terms which are not normal ordered

$$\frac{1}{2}(\sum_{n=1}^{m} k\cdot\alpha_{m-n} e^{in\tau}) V(k,\tau)$$

When performing the normal ordering in these terms, we get an extra commutator

$$\left[\frac{1}{2}\sum_{n=1}^{m} k\cdot\alpha_{m-n} e^{in\tau}, V(k,\tau)\right] = \frac{1}{2} m k^2 e^{im\tau} V(k,\tau)$$

So, finally we obtain

$$[L_m, V(k,\tau)] = e^{im\tau}\left(-i\frac{\partial}{\partial\tau} + \frac{1}{2} m k^2\right) V(k,\tau)$$

Therefore, the conformal weight of the operator $V(k,\tau)$ is

$$\Delta = \frac{k^2}{2}$$

The operator (5) is a vertex operator for the open string tachyon. The vertex operator for the massless vector field, can be constructed in a similar way and it has the form

$$V(k,\tau) =: e_\mu \; \frac{dX^\mu(0,\tau)}{d\tau} \; e^{ik\cdot X(0,\tau)} :$$

and the polarization vector obeys the transversality condition $\partial_\mu e^\mu = 0$. Similar computations for the closed string give

$$\Delta = \frac{k^2}{4}$$

In general closed string vertex operators have a form

$$A_{m_1\mu_1,,\dots,m_k\mu_k;n_1\nu_1,,\dots,n_l\nu_l} =: \partial^{m_1}_{z_1} X_{\mu_1} \; \dots\partial^{m_k}_{z_k} X_{\mu_k} \; \partial^{n_1}_{\bar z_1} X_{\mu_1} \; \dots\partial^{n_l}_{\bar z_l} X_{\mu_l} \; e^{ik\cdot X} : \tag{7}$$

Performing the computations similar to the ones we have done above, one can find that the conformal dimension of the operator ∂X^μ is equal to 1. Therefore the conformal dimension of the vertex operator (7) is equal to

$$\Delta = \frac{k^2}{4} + \sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j$$

We have the following correspondence between states and the operators (see the Lecture 3)

$$A_{m_1\mu_1,,\dots,m_k\mu_k;n_1\nu_1,,\dots,n_l\nu_l} \leftrightarrow \alpha^+_{m_1,\mu_1}\dots\alpha^+_{m_k,\mu_k}\bar\alpha^+_{n_1,\nu_1}\dots\bar\alpha^+_{n_l,\nu_l}|0,k\rangle$$

Generic vertex operators for the open string have the same form as (7), except now we have only one type of oscillators and the operators are taken at the points $\sigma = 0$ or $\sigma = \pi$. Therefore, for an Open String we have

$$\Delta = \frac{k^2}{2} + \sum_{i=1}^{k} m_i$$

From the requirements $\Delta = 2$ for the Closed String and $\Delta = 1$ for the Open String we can find masses (i.e., the value of k^2) for the corresponding string excitations.