LECTURE 6. CIRCLE COMPACTIFICATIONS. D-BRANES. PART I

1. T-DUALITY

1.1. Closed Strings. Let us consider a duality which is present in String Theory, and has no analogue in Field Theory.

Let us recall the expression for the coordinates of the Closed String

$$x^\mu(\tau,\sigma) = x_0^\mu + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \quad (oscillators)$$

where we take $-\infty < \tau < \infty$ and $0 \leq \sigma \leq 2\pi$.

The center of mass momentum is obtained by taking the derivative with respect to τ and then integrating over σ

$$p_0^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$$

and since the coordinates are periodic

$$x^\mu(\tau,\sigma+2\pi) = x^\mu(\tau,\sigma), \quad \to \alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p_0^\mu \tag{1}$$

Now, let us consider the situation, when one coordinate, say x^{25}, is compactified on a circle of radius R

$$x^{25}(\tau,\sigma) \sim x^{25}(\tau,\sigma) + 2\pi R \tag{2}$$

In order the string wave function to be single valued, the momentum p^{25}, should be quantized (because of the factor $e^{ip_0^{25}x_0^{25}}$ in the free string wave function), i.e.

$$p_0^{25} = \frac{n}{R}$$

This quantization condition, is valid for any system, not only for a string. We have

$$\alpha_0^{25} + \tilde{\alpha}_0^{25} = \frac{2n}{R}\sqrt{\frac{\alpha'}{2}} \tag{3}$$

Besides, the string can wrap around the compact dimension

$$x^{25}(\tau,\sigma) \sim x^{25}(\tau,\sigma) + 2\pi\omega R \tag{4}$$

This is a new boundary condition, which yields to the following expansion

$$x^{25}(\tau,\sigma) = x_0^{25} + \sqrt{\frac{\alpha'}{2}}(\alpha_0^{25} + \tilde{\alpha}_0^{25})\tau + \omega R\sigma + \quad (oscillators) \tag{5}$$

Comparing (1) and (5) we get

$$\alpha_0^{25} - \tilde{\alpha}_0^{25} = \omega R\sqrt{\frac{2}{\alpha'}} \tag{6}$$

From (3) and (6) it follows that

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}p_L, \quad p_L = \frac{n}{R} + \frac{\omega R}{\alpha'} \tag{7}$$

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}p_R, \quad p_R = \frac{n}{R} - \frac{\omega R}{\alpha'} \tag{8}$$

Let us consider the mass spectrum

$$m^2 = -p^\mu p_\mu, \quad \mu = 0, 1, ..., 24$$

We also have the Virasoro constraints

$$\frac{1}{2}\alpha_0^2 + N - 1 = \frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + \frac{1}{2}(\alpha_0^{25})^2 + N - 1 = 0$$

$$\frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N} - 1 = \frac{1}{2}\tilde{\alpha}_0^\mu \tilde{\alpha}_{0\mu} + \frac{1}{2}(\tilde{\alpha}_0^{25})^2 + \tilde{N} - 1 = 0$$

This means

$$-p^\mu p_\mu = \frac{4}{\alpha'}(N-1) + \frac{2}{\alpha'}(\alpha_0^{25})^2$$

So, finally

$$m^2 = \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) = \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\tilde{N}-1)$$

Or equivalently,

$$m^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{9}$$

States, which have $n \neq 0$ are usual Kaluza-Klein modes. States with $\omega \neq 0$ arise from the fact that the string has nonzero length, i.e., it is an extended object. The states with $\omega \neq 0$ have no analogue in the field theory. They are called winding modes. The states with $n \neq 0$ are called momentum modes. The usual "noncompact" modes are obtained by taking $n = \omega = 0$. In particular massless "noncompact" modes are obtained by taking $n = \omega = 0$ and $N = \tilde{N} = 1$.

Let us consider the limit $R \to \infty$. As one can see from (9), the states which have $\omega \neq 0$ become infinitely massive and decouple. The states with $\omega = 0, n \neq 0$ form a continuum. To summarize, the fields do not depend on the compactified coordinates and we effectively have a theory with one dimension less that the original one (that is what happens in Field Theory).

Now let us consider a limit $R \to 0$. In this limit states with $n = 0$ and $\omega \neq 0$ form a continuum. Therefore in this limit a compact dimension contributes to the mass spectrum ($\omega \neq 0$ means, that the string is wrapped around the extra dimension). This is a stringy effect. This is called decompactification. In the case of the Field Theory, we would have had the situation, when the lower dimensional fields simply do not depend on the extra dimension.

Let us notice, that (9) is invariant under interchange

$$n \leftrightarrow \omega, \quad R \leftrightarrow R' = \frac{\alpha'}{R}$$

which according to (7) and (8) means the following symmetry symmetry

$$\alpha_0^{25} \to \alpha_0^{25}, \quad \tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25},$$

This symmetry is called T duality. It means that we can compactify a closed string on a circle of the radius R and then obtain an equivalent theory by replacing the radius R with $\frac{\alpha'}{R}$ and interchanging momentum and winding modes $n \leftrightarrow \omega$. This process is called T dualization. It means the following transformations for the string coordinates

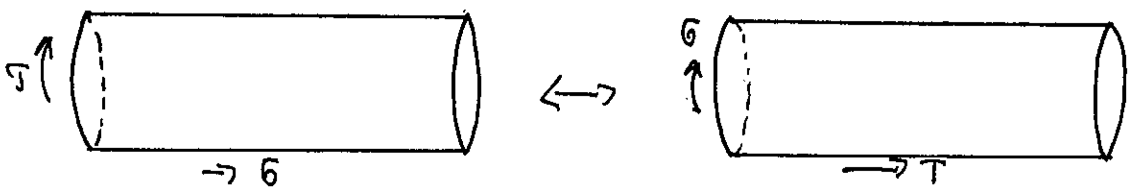
$$x_L^{25}(\tau+\sigma) \to x_L^{25}(\tau+\sigma), \quad x_R^{25}(\tau-\sigma) \to -x_R^{25}(\tau-\sigma)$$

Therefore we can always consider a dual coordinate

$$\bar{x}^{25} = x_L^{25} - x_R^{25}$$

because this expression similarly to $x_L^{25} + x_R^{25}$ is also a solution of the equations of motion. The only difference between these theories is a sector of zero modes, where we should perform the transformation $R \to \frac{\alpha'}{R}$. Therefore T duality is a symmetry of the interacting theory, since all vertex operators[1] are functions of x^μ coordinates and of their derivatives.

1.2. **Open Strings.** Let us consider now the case of an open string. Since it can no wrap around the compact dimension, there is no winding number ω. Therefore, in the limit $R \to 0$ the open string behaves as a Field Theory does. That means, Kaluza-Klein modes $n \neq 0$ are infinitely massive. But now, unlike the situation for the closed string we do not have a continuum of states $(n = 0, \omega \neq 0)$. Therefore we have a problem in the theory: Open string theory is not consistent without closed string. Indeed the loop diagram for an open string is equivalent to the tree level diagram for the closed string



As we noticed above, the open string lives in 25 dimensions in the limit $R \to 0$ -there is no decompactification. On the other hand, the closed string lives in 26 dimensions in the same limit- there is a decompactification. The solution for this discrepancy is the following. The open string lives in 26 dimensions, but its endpoints are confined to a 25 dimensional hyperplane. Therefore both open and closed strings live in 26 dimensions, but the open string endpoints are attached to a 25 dimensional hyperplane.

Recall that

$$x^\mu(\tau \pm \sigma) = \frac{x_0^\mu}{2} \pm \frac{\bar{x}_0^\mu}{2} + \sqrt{\frac{\alpha'}{2}}(\tau \pm \sigma)\alpha_0^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau \pm \sigma)}$$

[1]Interactions between string states are expressed via the corresponding vertex operators, see next Lectures.

where $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ and $\bar{x}_0^\mu$ is an integration constant. The solution of the equation motion for the open string is

$$x^\mu(\tau,\sigma) = x^\mu(\tau+\sigma) + x^\mu(\tau-\sigma) = x_0^\mu + \alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma)$$

Now, let us compactify the 25-th coordinate on a circle with radius R

$$x^{25}(\tau,\sigma) \sim x^{25}(\tau,\sigma) + 2\pi R,$$

Therefore

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}$$

Consider a T dual coordinate

$$\bar{x}^{25} = x^{25}(\tau+\sigma) - x^{25}(\tau-\sigma) = x_0^{25\prime} + \frac{2\alpha' n\sigma}{R} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\sin(n\sigma)$$

Since the oscillator-independent part does not contain τ, the corresponding momentum will be zero. Therefore, the dual string has no momentum in this direction. And since $\sin(n\sigma) = 0$ for $\sigma = 0, \pi$, the endpoints of the string are not moving along the 25-th dimension. Therefore, the string coordinate x^{25} satisfies Neumann boundary conditions

$$\partial_\sigma x^{25}|_{\sigma=0,\pi} = 0$$

and the dual coordinates $\bar{x}^{25}$ are satisfying Dirichlet boundary conditions.

$$\partial_\tau \bar{x}^{25}|_{\sigma=0,\pi} = 0$$

Further,

$$\bar{x}^{25}(\tau,\pi) - \bar{x}^{25}(\tau,0) = \frac{2\pi\alpha' n}{R} = 2\pi n R' \tag{10}$$

That means that the dual coordinates of the endpoints of the open strings are identical up the periodicity of the dual coordinate. The equation (10) thus represents an open string analogue of the condition (4).

The 24 dimensional hyperplane on which the endpoints of the open string are stuck is called $D24$ brane. Therefore in the T-dual picture one sees the opens string whose endpoints are attached to the $D24$ brane.

Similarly, if we dualise m coordinates, then we get Dp brane, where $p = 25 - m$.