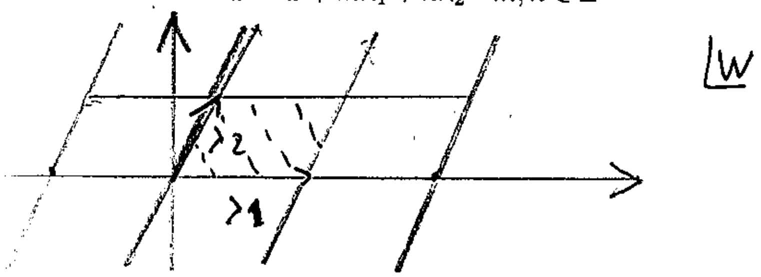
LECTURE 5. PARTITION FUNCTION ON TORUS. PART II

However, we have a certain gauge invariance, which we should fix, when performing integration over τ.

We can obtain a torus by identifying the points on a complex plane as

$$w \sim w + m\lambda_1 + n\lambda_2 \quad m, n \in \mathbb{Z}$$



Here λ_1 and λ_2 are two complex numbers. Their ratio is called the complex structure of the torus

$$\tau = \frac{\lambda_1}{\lambda_2}$$

We can choose the pair λ_1 and λ_2 differently and it still can describe the same lattice. If two different choices describe the same lattice then we have

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}$$

In order to the inverse matrix to have an integer entries we impose

$$|ad - bc| = 1$$

Apparently the lattice defined by (λ_1, λ_2) is the same as $(-\lambda_1, -\lambda_2)$. Therefore we have to divide by the action of the group Z_2. Obtained transformations form a group which is called $PSL(2, \mathbb{Z})$ and is known as a modular group of a torus. The group is formed by 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

whose elements are integer numbers and the determinant is equal to 1. We can choose $\lambda_1 = 1$ and $\lambda_2 = \tau$, therefore the modular transformations the parameter τ transforms as

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Let us take coordinates on a torus as $0 \leq \sigma_{1,2} \leq 1$. The metric on a torus can be written as

$$g_{ab} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau| \end{pmatrix}.$$

The metric is parametrized by one complex variable τ, which is called a modular parameter. It parametrizes unequivalent tori. The line element on a torus is

$$ds^2 = g_{ab} d\sigma_a d\sigma_b = \frac{dw d\bar{w}}{\tau_2}$$

where $w = \sigma_1 + \tau\sigma_2$. The coordinates $\sigma_{1,2}$ are periodic, which translates into periodicity of the coordinates w:

$$\sigma_1 \sim \sigma_1 + 1 \to w \sim w + 1$$
$$\sigma_2 \sim \sigma_2 + 1 \to w \sim w + \tau$$

Tori have diffeomorphisms that can not be made an identity by continuous deformations. They are expressed as

$$(\sigma_1, \sigma_2) = (a\sigma_1 + b\sigma_2, c\sigma_1 + d\sigma_2)$$

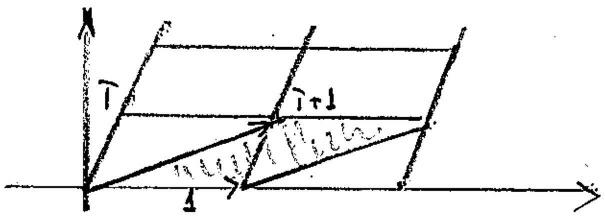
The constants a, b, c and d are integer.

1. Particular cases

1. $a = 1$, $b = 1$, $c = 0$, $d = 1$. We have

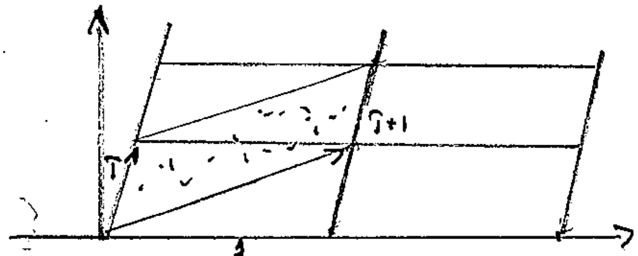
$$\tau \to \tau + 1, \quad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

These transformations are called T-transformations.



Apparently, the new torus is equivalent to the old one, due to the periodicity of σ_1 and σ_2. On the other side the new torus corresponds to $\tau + 1$. Therefore, we have invariance under $\tau \to \tau + 1$.

2. Let us consider another choice of the defining parallelogram with $\lambda_1 = 1+\tau$, and $\lambda_2 = \tau$



We can rescale the both sides of the parallelogram by the factor $\frac{1}{\tau+1}$ in order to bring it to the original form. Then we get a torus with the modulus $\frac{\tau}{\tau+1}$. These are called U transformations

$$\tau \to \frac{\tau}{\tau+1}$$

however it is more convenient to consider S transformations defined as

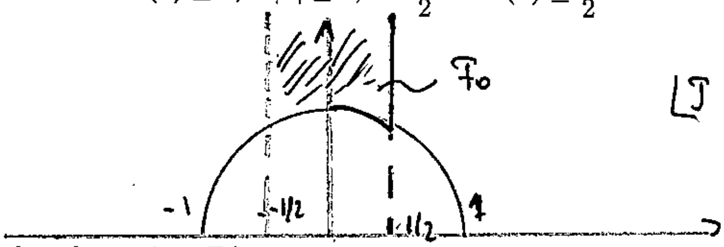
$$\tau \to -\frac{1}{\tau}, \quad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which corresponds to $a = 0$, $b = 1$, $c = -1$, $d = 0$. The S and T transformations generate the modular group. We also have

$$S^2 = (ST)^3 = 1, \quad S = UT^{-1}U$$

Since we established the invariance of the integral under $PSL(2,\mathbb{Z})$ group, the next step is to define the integration area with respect to the parameter τ. Let us consider the following domain on the complex half-plane

$$Im(\tau) \geq 0, \quad |\tau| \geq 1, \quad -\frac{1}{2} < Re(\tau) \leq \frac{1}{2}$$



This domain (often denoted as $\mathcal{F}_0$) is called the fundamental domain of the torus. One can prove two important statements:

- Any point on the complex half-plane, which is outside $\mathcal{F}_0$ can be brought inside $\mathcal{F}_0$ using the modular group transformations.
- For any point z_0 which is inside $\mathcal{F}_0$, and for any element $g \neq 1$ of the modular group, the point gz_0 is outside $\mathcal{F}_0$.

To obtain the moduli space of the torus we impose the further identifications: The boundaries with $Im(\tau) = -\frac{1}{2}$ and $Im(\tau) = \frac{1}{2}$ are identified under the equivalence $\tau \sim \tau + 1$. The points with $|\tau| = 1$ are identified because of $\tau \sim -\frac{1}{\tau}$. This means that the moduli space of the tori (moduli are the parameters of a Riemann surface, in general) is the folding of the fundamental domain along the imaginary axis and gluing the boundaries.

Therefore in the previous lecture we almost obtained a correct partition function on the torus. We did not have a correct integration domain with respect to τ. We have derived a correct integration domain in the present lecture. It is a fundamental domain of the torus.

1.1. **Partition Function. Again.** Let us recall, that for path integrals in Quantum Mechanics we have

$$Z = \int \mathbb{D}q\, e^{-S(q)} = Tr e^{-\beta H}$$

Here $q(t)$ is a coordinate and we consider the time interval $0 \leq t \leq \beta$ and $q(0) = q(\beta)$. this can be generalized to the case of the closed string on the torus. Consider a point on the string which we put on the real axis, going upwards in time[1] $2\pi\tau_2$. This is generated by the Hamiltonian $H = L_0 + \tilde{L}_0 - 2$. At the same time this point undergoes a shift $2\pi\tau_1$ generated by the momentum $P = L_0 - \tilde{L}_0$. Putting this together, we get

$$Z = \int_{\mathcal{F}_0} d^2\tau \frac{1}{q\bar{q}} \int d^{24}p e^{-\frac{\pi\tau_2 p^2}{2}} Tr(q^N \bar{q}^{\tilde{N}})$$

Let us see, how the vacuum energy transforms under the modular transformations. Notice, that under the modular transformations the Dedekind function transforms as

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$$

Under T transformations the measure $\frac{d\tau\, d\bar{\tau}}{\tau_2^2}$ is invariant and $\eta(\tau)\eta(\bar{\tau})$ is invariant. Therefore $\Gamma_{st.}$ is invariant.

Under S transformations

$$\tau_2' = \frac{\tau_2}{\tau\bar{\tau}}, \quad d\tau' d\bar{\tau}' = \frac{d\tau d\bar{\tau}}{(\tau\bar{\tau})^2}$$

which means that the measure is invariant. The expression $\sqrt{\tau_2}\eta(\tau)\eta(\bar{\tau})$ is invariant as well. Therefore, the vacuum energy is invariant under S- transformations as well.

1.2. **Closed Bosonic string spectrum.** Using

$$\frac{1}{\eta^n(q)} = q^{-\frac{n}{24}}(1 + nq + ...)$$

one can evaluate

$$\frac{1}{[\eta(\tau)\eta(\bar{\tau})]^{24}} = \frac{1}{q\bar{q}}(1 + 24(q+\bar{q}) + 24 \times 24 q\bar{q} + ...)$$

from this expression one can read the spectrum of the closed bosonic string. The rules are:

- A power of q or $\bar{q}$ gives as the corresponding mass (i.e. the eugenvalue of N or $\tilde{N}$).
- The coefficient in front of q or $\bar{q}$ gives us the number of physical degrees of freedom.

[1] Note, we restored the factor of 2π

For example:

The first term $\frac{1}{q\bar{q}}$ has $N = \tilde{N} = -1$, therefore it is a tachyon. The coefficient is 1, i.e., it is a scalar.

The term $\frac{24q}{q\bar{q}}$ is not a physical state. Indeed, it has $N = -1$ and $\tilde{N} = 0$ and it does not satisfy the level matching condition $N = \tilde{N}$.

The expression $\frac{24\times 24 q\bar{q}}{q\bar{q}}$ means that we have 24×24 degrees of freedom on the zero mass level. They correspond to the graviton $g_{(ij)}$, to the antisymmetric field $B_{[ij]}$ and to the dilaton.

Therefore we have built a modular invariant expression for the vacuum energy for the closed bosonic string. This expression is also called a partition function.