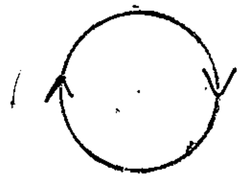
LECTURE 4. PARTITION FUNCTION ON TORUS. PART I

1. Vacuum energy

1.1. Definition. In Quantum Field Theory the vacuum energy is expressed via Feynman diagrams of the type



It is a function of the mass of the field. In String Theory we have an infinite number of the fields and one can read the spectrum of the string from the vacuum energy. When one compactifies a string to lower dimensions, and computes the vacuum energy, the vacuum energy will be defined by the relevant geometry. Therefore the in String Theory the vacuum energy allows one to obtain the spectrum of various string-derived physical systems.

1.2. Scalar field. Let us start from the field theory and consider a case of a scalar field with mass M in the Euclidean space . We have a functional integral

$$Z(J) = \int \mathbb{D}\phi \, e^{-\int d^Dx[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)+\frac{M^2}{2}\phi^2 - J\phi]} = e^{-W[J]}$$

The effective action $\Gamma[\phi]$ is defined via Legendre transformation

$$\Gamma[\phi] = W[J] + \int d^DxJ\phi$$

Since

$$\frac{\delta\Gamma}{\delta\phi_{cl.}} = J$$

The effective action $\Gamma[\phi_{cl.}]$ has an extremum at $J = 0$. The value of Γ at the extremum is called a vacuum energy

$$e^{-\Gamma} = \int \mathbb{D}\phi \, e^{-\int d^Dx[-\frac{1}{2}\phi\Box\phi+\frac{M^2}{2}\phi^2]}$$

Let us compute it using the Gaussian integration

$$I = \int d^Dx \, e^{-x_iA_{ij}x_j} = (detA)^{-\frac{1}{2}}\pi^{\frac{D}{2}}$$

Therefore

$$\Gamma = \frac{1}{2}\ln(det(-\Box + M^2))$$

Let us use the identities

$$\ln(detA) = tr\ln(A)$$

and

$$\ln a = -\int_{\epsilon}^{\infty} \frac{dt}{t} e^{-ta}$$

where t is a Schwinger parameter and ϵ is an (ultraviolet) cut-off. Therefore we get

$$\Gamma = -\frac{1}{2}\int_{\epsilon}^{\infty} \frac{dt}{t} tr(e^{-t(-\Box+M^2)})$$

The trace over an operator is understood as a sum of its eigenvalues. The eigenvalues of $\Box$ are $-p^2$. For continuous values of the momentum one replaces the sum with the integral

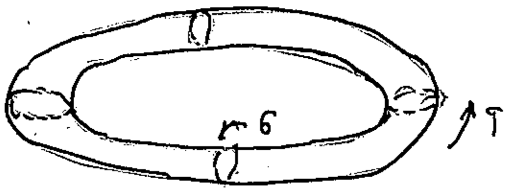
$$\Gamma = -\frac{V}{2}\int_{\epsilon}^{\infty} \frac{dt}{t} e^{-tM^2} \int \frac{d^D p}{(2\pi)^D} e^{-tp^2}$$

where V is a volume of the space-time. Finally, performing the Gaussian integral with respect to p we finally get

$$\Gamma = -\frac{V}{2(4\pi)^{\frac{D}{2}}}\int_{\epsilon}^{\infty} \frac{dt}{t^{\frac{D}{2}+1}} e^{-tM^2} \tag{1}$$

This is a vacuum energy for one scalar.

1.3. **Closed String.** In case of the closed string the vacuum diagram naturally defines a torus.



We are going to generalize the expression for the vacuum energy (1), which we obtained for the massive scalar field, to the case of the closed bosonic string[1]. Let us now compute $tr(e^{-tM^2})$. Now we have an infinite number of states and the bosonic string lives in the critical dimension $D = 26$. Let us recall that

$$L_0 = \frac{\alpha'}{4}p^2 + \frac{1}{2}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n, \quad \tilde{L}_0 = \frac{\alpha'}{4}p^2 + \frac{1}{2}\sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n,$$

We also have the level -matching condition $N = \tilde{N}$, which we incorporate using the delta function

$$\delta(N-\tilde{N}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} ds\, e^{2\pi i s(N-\tilde{N})}$$

Collecting everything together we get

$$\Gamma_{str.} = -\frac{V}{2}\int \frac{d^D p}{(2\pi)^D} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{\epsilon}^{\infty} \frac{dt}{t^{14}} tr\left(e^{2\pi i s(N-\tilde{N})} e^{-\frac{2}{\alpha'}(N+\tilde{N}-2)t}\right)$$

[1]More rigorous argument will be given in the next Lecture

Let us define a complex Schwinger parameter

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi}$$

and introduce

$$q = e^{2\pi i\tau}, \quad \bar{q} = e^{-2\pi i\tau}$$

Then we get

$$\Gamma_{str.} = -\frac{V}{(4\pi)^{13}}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{14}} tr(q^{N-1}\bar{q}^{\tilde{N}-1})$$

Now let us evaluate the trace. Recall, that in the light cone gauge

$$N = \sum_{n=1}^{\infty}\sum_{i=1}^{24} \alpha_{-n}^i \alpha_n^i$$

Therefore we have

$$tr(q^N) = tr(q^{\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n})^{24}$$

and since $[\alpha_{-n}^i, \alpha_n^j] = n\delta^{ij}$ we have for each oscillator

$$1 + q^n + q^{2n} + \ldots = \frac{1}{1-q^n}$$

Finally,

$$tr(q^N) = \left(\frac{1}{1-q^n}\right)^{24}$$

Let us introduce the Dedekind function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Then, finally (up to an overall constant)

$$\Gamma_{st.} = \int \frac{d\tau d\bar{\tau}}{\tau_2^2}\frac{1}{\tau_2^{12}}\frac{1}{|\eta(\tau)|^{48}} \tag{2}$$

The integration domain is