LECTURE 3. BOSONIC STRING. QUANTIZATION

1. Constraints

Let us do the cannonical Quantization of the closed bosonic string. Consider commutation relations

$$[x^\mu(\tau,\sigma),\ \frac{1}{2\pi\alpha'}\dot{x}^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$$

Using

$$\delta(\sigma-\sigma') = \sum_n \frac{e^{2ni(\sigma-\sigma')}}{\pi}$$

we can obtain from this commutator commutation relations for the oscillators

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}, \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}$$

$$[x^\mu_0, p^\nu_0] = i\eta^{\mu\nu}, \quad [\tilde{\alpha}^\mu_m, \alpha^\nu_n] = 0$$

In order to get canonical commutation relations, we can redefine the oscillators as

$$\alpha^\mu_m = \sqrt{m}\,\alpha^\mu_m, \quad \alpha^\mu_{-m} = \sqrt{m}\,\alpha^{\mu,+}_m, \quad m>0$$

$$\tilde{\alpha}^\mu_m = \sqrt{m}\,\tilde{\alpha}^\mu_m, \quad \tilde{\alpha}^\mu_{-m} = \sqrt{m}\,\tilde{\alpha}^{\mu,+}_m, \quad m>0$$

$$[\alpha^\mu_m, \alpha^{\nu,+}_n] = \eta^{\mu\nu}\delta_{m,n}, \quad [\tilde{\alpha}^\mu_m, \tilde{\alpha}^{\nu,+}_n] = \eta^{\mu\nu}\delta_{m,n}$$

Therefore α^μ_m and $\tilde{\alpha}^\mu_m$ are annihilation operators, whereas $\alpha^{\mu,+}_m$ and $\tilde{\alpha}^{\mu,+}_m$ are creation operators.

As in the Quantum Field Theory, the oscillators $\alpha^{0,+}_m$ and $\tilde{\alpha}^{0,+}_m$ describe the states with a negative norm

$$\alpha^{0,+}_m|0\rangle, \quad \langle 0|\alpha^0_m\,\alpha^{0,+}_m|0\rangle < 0$$

Recall, that we had

$$T_{++}(\xi^+) = \frac{1}{2}(\partial_+ x^\mu_L)(\partial_+ x_{L,\mu}), \quad T_{--}(\xi^-) = \frac{1}{2}(\partial_- x^\mu_R)(\partial_- x_{R,\mu}), \quad T_{+-} = 0$$

Using the explicit form for x^μ_L and x^μ_R we get

$$T_{++} = \sum_{n=-\infty}^{\infty} \tilde{L}_n e^{2in\xi^+}, \quad T_{--} = \sum_{n=-\infty}^{\infty} L_n e^{2in\xi^-}$$

where

$$\tilde{L}_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \tilde{\alpha}^\mu_{n-m}\cdot\tilde{\alpha}^\mu_m, \quad L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^\mu_{n-m}\cdot\alpha^\mu_m \qquad (1)$$

These expressions are correct for $n \neq 0$, since the oscillators in each term of these expressions commute among each other. The exception are L_0 and $\tilde{L}_0$ since α^μ_{-m} ($\tilde{\alpha}^\mu_{-m}$) do not commute with α_m^μ ($\tilde{\alpha}_m^\mu$). Recall the Quantum Mechanics. If we have a classical expression which contains a product of x and p, (for example xp), then Quantum Mechanically we should write $\hat{x}\hat{p}+c$ where c is a constant. Here we have the same situation: define

$$\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^\mu \cdot \tilde{\alpha}_0^\mu + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^\mu \cdot \tilde{\alpha}_m^\mu - a, \quad L_0 = \frac{1}{2}\alpha_0^\mu \cdot \alpha_0^\mu + \sum_{m=1}^{\infty} \alpha_{-m}^\mu \cdot \alpha_m^\mu - a \tag{2}$$

where a is a constant to be determined.

Let us notice also, that $L_0 + \tilde{L}_0$ is a Hamiltonian H for the Closed String. Recall that the Hamiltonian density $\overline{H}$ is

$$\overline{H} = \dot{x}^\mu \Pi_\mu - L = \frac{1}{4\pi\alpha'}\left((\partial_\sigma x^\mu)(\partial_\sigma x_\mu) + (\partial_\tau x^\mu)(\partial_\tau x_\mu)\right)$$

and

$$H = \int_0^\pi d\sigma \overline{H}$$

Because x_0^μ and p_0^μ satisfy the usual Heisenberg commutation relations, we can take a state vector as

$$|k,0\rangle = e^{ikx}|0\rangle$$

i.e., the Hilbert space for the closed string is built using the following vacuum

$$p_0^\mu|k,0\rangle = k^\mu|k,0\rangle,$$

$$\alpha_n^\mu|k,0\rangle = \tilde{\alpha}_n^\mu|k,0\rangle = 0, \quad for \quad n>0$$

Let us recall the Dirac quantization procedure. In general we have a Lagrangian which is a function of q_i and $\dot{q}_i$. Let us perform the Legandre transformation $H = p_i\dot{q} - L$ i.e., move from the Lagrangian to Hamiltonian description. In this process we should express the momenta p_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

in terms of q_i and $\dot{q}_i$. In this process sometimes we can find that some function(s) of the coordinates and momenta is zero

$$F_m(p^j, q^j) = 0$$

These functions are called primary constraints. When performing the Quantization, we should require that the constraints annihilate the physical states

$$F_m(\hat{p}^j, \hat{q}^j)|phys\rangle = 0$$

If the constraints satisfy an algebra with respect to the commutation relations $[F_m, F_n] = C_{mn}^k F_k$, they are called the first class constraints. Otherwise they are called the second class constraints. The quantization of the systems which contain only first class constraints is usually easier, than for the ones which contains the second class constraints. We can also divide the constraints into two sets. The constraints from the first set satisfy

$$0 = \langle phys|F_{m_1}, \quad m_1 = 1,..,k$$

and

$$F_{m_2}|phys\rangle = 0 \quad m_2 = k+1, .., m$$

like it happens, for example, in the Electrodynamics.

In our case we have Virasoro constraints. One can show that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

Here the constant c equals to the number of space-time dimensions d. It is a central charge and represents so called conformal anomaly. That means, the conformal symmetry which is present at the classical level, is generally broken by quantum corrections.

Note: Two dimensional conformal symmetry is infinite dimensional. It has an infinite number of generators. They are called Virasoro operators.

The quantization conditions are

$$L_0|phys\rangle = 0, \quad \tilde{L}_0|phys\rangle = 0 \tag{3}$$

$$L_m|phys\rangle = 0, \quad \tilde{L}_m|phys\rangle = 0, \tag{4}$$

for $m > 0$ and Virasoro generators are defined in (1) and (2).

From (3) we get

$$(L_0 - \tilde{L}_0)|phys\rangle = 0, \quad and \quad (L_0 + \tilde{L}_0)|phys\rangle = 0 \tag{5}$$

From (2) it is clear that masses of the states have the form

$$M^2 = \frac{2}{\alpha'}\left(\sum_{m=1}^{\infty}(\tilde{\alpha}^{\mu}_{-m}\cdot\tilde{\alpha}^{\mu}_{m} + \alpha^{\mu}_{-m}\cdot\alpha^{\mu}_{m}) - 2a\right)$$

Let us introduce a notation

$$N = \sum_{m=1}^{\infty}\tilde{\alpha}^{\mu}_{-m}\cdot\tilde{\alpha}^{\mu}_{m}, \quad \tilde{N} = \sum_{m=1}^{\infty}\alpha^{\mu}_{-m}\cdot\alpha^{\mu}_{m}$$

Then the first equation in (5) will give us the so called "level matching condition"

$$N = \tilde{N}$$

Before we move to the description of the quantum states, let us consider the case of an Open String. Here we have oscillators of only one type. The zero mode is

$$\alpha^{\mu}_0 = \sqrt{2\alpha'}\, p^{\mu}_0$$

The quantization conditions have the form

$$L_0|phys\rangle = 0, \tag{6}$$

$$L_m|phys\rangle = 0, \tag{7}$$

and the masses of the states are

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m=1}^{\infty} \alpha^{\mu}_{-m} \cdot \alpha^{\mu}_{m} - a\right)$$

Let us consider the case, when the physical state does not contain any oscillator (the zeroth level) i.e., $N = 0$. Then for an Open String we have

$$M^2 = -\frac{a}{\alpha'}$$

and for the Closed String we have ($N = \tilde{N} = 0$)

$$M^2 = -\frac{4a}{\alpha'}$$

If $a > 0$ then for both Open and Closed Strings on the zeroth level we have a particle with a negative mass square i.e., tachyons.

Let us consider the case when the vacuum for an Open String contains one oscillator $\tilde{N} = 1$. Then the corresponding state will have the form

$$|phys\rangle = e_{\mu}(k)\alpha^{\mu}_{-1}|k, 0\rangle$$

The Virasoro conditions will give us the following equations. From (6) we get

$$M^2 = \frac{1-a}{\alpha'}$$

and from (7) we get

$$k^{\mu}e_{\mu}(k) = 0$$

Now let us determine the constant a. To this end let us move to the light cone coordinates in the space-time. Then one can choose a gauge

$$x^+ = x_0^+ + (2\alpha')p^+\tau$$

and decompose the index μ, where $\mu = 0, ...d-1$ into $(+, -, i)$ where $i = 1, ..., d-2$. Then from the Virasoro constraints $T_{ab} = 0$ one can express the α^+_m and α^-_m oscillators in terms of α^i_m oscillators.

The mass for the Closed String states is

$$M^2 = \frac{2}{\alpha'}\left(\sum_{m\neq 0}^{\infty} (\tilde{\alpha}^i_{-m} \cdot \tilde{\alpha}^i_m + \alpha^i_{-m} \cdot \alpha^i_m)\right)$$

and for the Open String

$$M^2 = \frac{1}{\alpha'}\left(\sum_{m\neq 0}^{\infty} \alpha^i_{-m} \cdot \alpha^i_m\right)$$

We have

$$\sum_{m\neq 0}^{\infty} (\tilde{\alpha}^i_{-m} \cdot \tilde{\alpha}^i_m) = 2\sum_{m=1}^{\infty} (\tilde{\alpha}^i_{-m} \cdot \tilde{\alpha}^i_m) + \sum_{m=1}^{\infty} [\tilde{\alpha}^i_{-m}, \tilde{\alpha}^i_m] = 2\sum_{m=1}^{\infty} (\tilde{\alpha}^{i+}_m \cdot \tilde{\alpha}^i_m) + (d-2)\sum_{m=1}^{\infty} m$$

The first term in the r.h.s is a number, whereas the second term is divergent and needs a regularization. Let us regularize it as

$$\sum_{m=1}^{\infty} m e^{-\epsilon m} = -\frac{d}{d\epsilon}\sum_{m=1}^{\infty} e^{-\epsilon m} = -\frac{d}{d\epsilon}\frac{1}{1-e^{-\epsilon}} = \frac{d}{d\epsilon}\left(\frac{1}{\epsilon}+\frac{1}{2}+\frac{1}{12}\epsilon+...\right) = \frac{1}{\epsilon^2}-\frac{1}{12}+...$$

The rest of the terms in the r.h.s vanish when $\epsilon \to 0$.

Therefore we drop the divergent term (regularization) and obtain for the mass for the Closed String states

$$M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - \frac{d-2}{12}\right) \tag{8}$$

and for the Open String

$$M^2 = \frac{1}{\alpha'}\left(N - \frac{d-2}{24}\right) \tag{9}$$

Let us go back to the covariant quantization. In d dimensions a massless vector has $d-2$ polarizations, whereas a massive vector has $d-1$ polarizations. From the light cone quantization it follows, that we have $d-2$ degrees of freedom, i.e., we must have a massless vector. This means

$$a = 1$$

Then from (8) and (9) we obtain

$$d = 26$$

Twenty six dimensions are called the critical dimensions for the bosonic string. Classically a bosonic string is well defined in any dimensions, but the quantization is consistent only in $d = 26$.

Let us notice that at the first excited level ($N = 1$) one can have a state which has a form

$$|\psi\rangle = L_{-1}|k,0\rangle$$

This state is orthogonal to all physical states

$$\langle phys|L_{-1}|k,0\rangle = \langle 0,k|L_{-1}|k,0\rangle = 0$$

These states should be removed from the spectrum. In other words, physical states are defined as

$$|phys\rangle \sim |phys\rangle + L_{-m}|\Lambda\rangle, \quad for \quad m > 0$$

States, which have the form $L_{-m}|\Lambda\rangle$ for $m > 0$, are called spurious states.

Therefore the Open String describes a massless vector field - a photon at the level $N = 1$. The spurion describes a gauge degree of freedom (Lorentz gauge)

$$A_\mu \sim A_\mu + \partial_\mu\Lambda, \quad \Box\Lambda = 0$$

since $\partial^\mu A_\mu = 0$, according to the Virasoro conditions.

For the Closed String at the first level $N = \tilde{N} = 1$ we have a state

$$X_{ij}(k)\,\tilde{\alpha}_{-1}^{i,+}\,\alpha_{-1}^{j,+}|0,k\rangle$$

From (8) it follows, that the mass of the field $X_{ij}(k)$ is zero. Recall that in d dimensions the massless fields are characterized by the representations of $SO(d-2)$. This means that the second rank tensor X_{ij} should be decomposed according to the representations of the group $SO(d-2)$. We obtain

$$g_{(ij)}, \quad B_{[ij]}, \quad and \quad Tr(X_{ij}) \equiv \phi$$

We have a symmetrical field g_{ij} - a graviton, anti- simmetrical field B_{ij} and a scalar ϕ, called a dilaton.

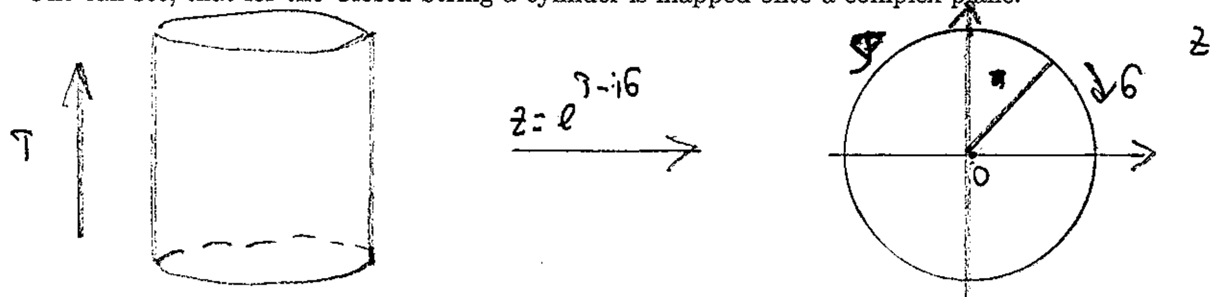
Therefore at the massless level the Open String describes the Maxwell field, while the Closed String describes Gravity. At the higher mass levels both Open and Closed Strings contain infinite towers of massive fields with masses linearly growing with the spins.

2. States and Operators

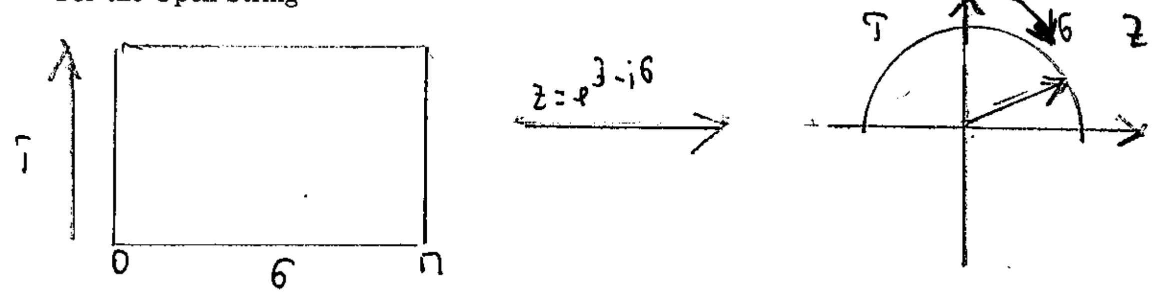
Let us move to the Euclidean signature on the world -sheet. To this end we replace $\tau \to i\tau$. Let us define the complex coordinate

$$z = e^{\tau - i\sigma}$$

One can see, that for the Closed String a cylinder is mapped onto a complex plane.



For the Open String



a strip is mapped onto an upper half-plane. For the energy momentum tensor we have a Laurent series

$$T_{++}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

For the String coordinates we get

$$x_L^\mu(z) = \frac{1}{2}x_0^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha_0^\mu \ln z + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu z^{-n}$$

$$x_R^\mu(\bar{z}) = \frac{1}{2}x_0^\mu - i\sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n}$$

Apparently

$$\partial_z x_L^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\alpha_n^\mu z^{-n-1}$$

$$\partial_{\bar{z}} x_R^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\alpha_n^\mu \bar{z}^{-n-1}$$

From these equations we can express the oscillators in terms of x^μ. For example for the Closed String

$$\alpha_{-n}^\mu = \sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi}z^{-n}\partial_z x_L^\mu(z), \quad \tilde{\alpha}_{-n}^\mu = \sqrt{\frac{2}{\alpha'}}\oint\frac{d\bar{z}}{2\pi}\bar{z}^{-n}\partial_z x_R^\mu(\bar{z}),$$

Therefore the oscillators α_{-1}^μ are obtained by taking the residue $\partial_z x_L^\mu(0)$, and the higher modes α_{-n}^μ are obtained in terms of $\partial_z^n x_L^\mu(0)$. In other words, we insert the operator into the point $z = 0$ and then take the contour integral.

The state which does not contain any oscillator i.e., the tachyon is described by the vertex operator

$$|0, k\rangle \to \int dz : e^{ikx} :$$

The symbol :: means the normal ordering: since the exponential contains both creation and annihilation operators we should write creation operators to the left of the annihilation operators, when expanding the exponential in terms of the oscillators.

For the Closed String at the first level $N = \tilde{N} = 1$ we have

$$X_{\mu\nu}\,\alpha_{-1}^\mu\,\alpha_{-1}^\nu|k, 0\rangle \to \int d^2z\, X_{\mu\nu} : \partial_z x^\mu\, \partial_{\bar{z}} x^\nu e^{ikx} :$$

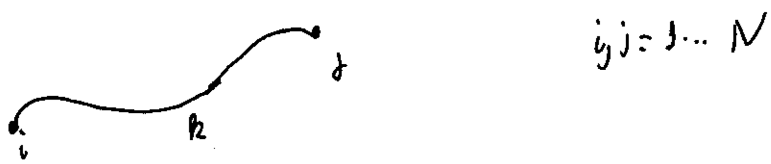
For the Open String at the first level $N = 1$ (the photon) we have

$$e_\mu\,\alpha_{-1}^\mu|k, 0\rangle \to \int dl\, e_\mu : \partial_t x^\mu\, e^{ikx} :$$

where l is a coordinate along the real axis and ∂_t is a tangential derivative with respect to this axis.

3. Chan-Paton factors

Chan - Paton factors allow one to introduce a non-abelian gauge symmetry. Let us add indices $i, j = 1, .., N$ to the Open String endpoints



and require that the Hamiltonian which corresponds to this degrees of freedom is zero. This means the corresponding degrees of freedom are constant in time.

Therefore we have for an Open string state

$$|k, a\rangle = \sum_{i,j=1}^{N} |k, ij\rangle (T^a)_{ij}$$

where $(T^a)_{ij}$ is a basis for $N \times N$ matrices. These matrices are called Chan-Paton factors. The Open String vertex operators , which we introduced in the previous Section also get Chan Paton-factors. For example, for the vector field we have

$$V^{a,\mu} = \int dl\, (T^a)_{ij}\, e_\mu : \partial_t x^\mu\, e^{ikx} :$$

The Chan - Paton symmetry is global one from the point of view of the String world-sheet, but is a local symmetry from the point of view of the space-time. This is because we can perform independent transformations at different points $x^\mu(\sigma, \tau)$ of the space-time.