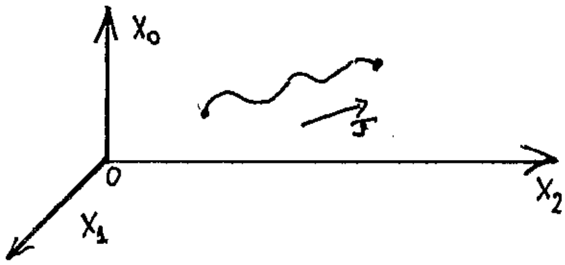
LECTURE 2. BOSONIC STRING. ACTION

1. Relativistic Particle

Let us consider a point particle with mass m and with spin equal to zero, propagating through a d- dimensional flat space-time. The metric is $\eta_{\mu\nu} = (-1, 1, 1..., 1)$.



The motion is described by a one dimensional worldline, which is parametrized by a proper time τ. The length of the worldline is

(1) $$dl = (-\eta_{\mu\nu} dx^{\mu} dx^{\nu})^{\frac{1}{2}} = (-ds^2)^{\frac{1}{2}}$$

The action describing the motion of the spinless relativistic particle is

$$S = -m \int d\tau \sqrt{-\dot{x}^{\mu}\dot{x}_{\mu}}, \quad where \quad \dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}$$

How do we know that this action describes a point particle? Let us consider equations of motion. Recall, that in general the momentum conjugate to the coordinate x is

$$p = \frac{\partial L}{\partial \dot{x}}$$

where L is the corresponding Lagrangian. In our case we have

$$p^{\mu} = \frac{m\dot{x}^{\mu}}{\sqrt{-\dot{x}^{\nu}\dot{x}_{\nu}}}$$

The equation motion with respect to x^{μ} gives us

$$\dot{p}^{\mu} = 0$$

which means that we have a free motion. Apparently,

$$p^{\mu}p_{\mu} = -m^2$$

and therefore the action described the motion of the free relativistic particle, with zero spin and mass equal to m.

The action is invariant under time reparametrizations

$$\tau \to \tau(\tau'), \quad \frac{d\tau}{d\tau'} > 0$$

because

$$d\tau\sqrt{-\dot{x}^\mu(\tau)\dot{x}_\mu(\tau)} = d\tau'\frac{d\tau}{d\tau'}\sqrt{-\frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau}\frac{dx_\mu}{d\tau'}\frac{d\tau'}{d\tau}} = d\tau'\sqrt{-\frac{dx^\mu}{d\tau'}\frac{dx_\mu}{d\tau'}}.$$

This means, that we have gauge invariance. We can choose a gauge, when $x^0 = \tau$.. In this gauge we have

$$S = -m\int d\tau\sqrt{1-v^2},$$

where

$$v_i = \frac{dx_i}{d\tau}, \quad i = 1,...,d-1$$

Equations of motion are

$$\dot{p}_i = 0, \quad where \quad p_i = \frac{mv_i}{\sqrt{1-v^2}}$$

The action that we are considering has two drawbacks. First, it is nonlinear, i.e., it contains a square root. Second, it is not good for description of massless particles. Instead, let us consider the action

$$S = \frac{1}{2}\int d\tau\left(\frac{1}{e}\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} - m^2 e\right) \tag{2}$$

where we have introduced a new field e, often called an einbein. This field is introduced in order to "remove" the square root and maintain the reparametrization invariance at the same time. Indeed the action is invariant under the infinitesimal transformations with the local parameter $\xi(\tau)$

$$\delta e = \frac{d(\chi e)}{d\tau}, \quad \delta x^\mu = \chi\frac{dx^\mu}{d\tau}$$

We can express the field e using its own equations of motion

$$\frac{1}{e^2}\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} + m^2 = 0$$

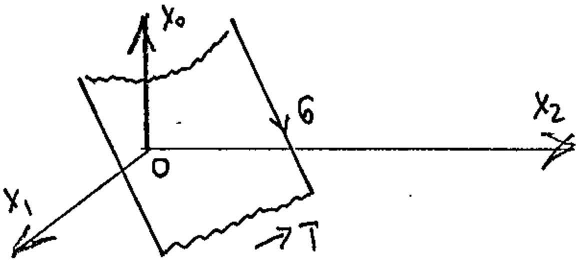
in terms of time derivatives of x^μ and put this expression back into the action. In this way we get the action we started with. It is easy to check that both actions (1) and (2) give us the same equations of motion.

Let us notice also that e has a meaning of the one dimensional metric on the world line of the particle. $g_{\tau\tau} = e^2$. Therefore we can writhe the action (2) as

$$S = \frac{1}{2}\int d\tau\sqrt{g}\left(g^{\tau\tau}\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} - m^2\right)$$

2. Action for a bosonic string

We considered the motion of a point particle with zero spin. A point particle is a zero-dimensional object and it spans one dimensional surface (a line). Now let us consider a motion of a two-dimensional object i.e., a motion of a string. It spans a two dimensional surface. For example for an open string we have



This surface can be parametrized by a proper time and by the length of the string $(\tau, \sigma) \equiv (\xi^0, \xi^1)$. Usually σ is taken to be $0 \leq \sigma \leq \pi$. Therefore, now we shall have $x^\mu(\tau, \sigma)$, whereas for a particle we had $x^\mu(\tau)$.

Similarly to what we had for a particle, let us write an action for a bosonic string (this action is called Polyakov action)

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-h} h^{ab} (\partial_a x^\mu)(\partial_b x^\nu) \eta_{\mu\nu} \tag{3}$$

where $\partial_a = \frac{\partial}{\partial \xi^a}$.

We have two different metrics: the metric h_{ab} on the two dimensional world-sheet, and the metric in the d dimensional space time $\eta_{\mu\nu}$. Therefore we have a $1+1$ dimensional Field Theory, where the coordinates x^μ are considered as fields.

We can also consider the action

$$S' = S + S_0, \quad where \quad S_0 = \int d^2\xi \sqrt{-h} R$$

where R is a Ricci scalar on a two dimensional world sheet. But in two dimensions Einstein equations are trivially satisfied

$$R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R = 0 \to h^{\mu\nu}(R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R) = 0$$

for any $h_{\mu\nu}$. The term S_0 is called a topological term. One can show that in two dimensions this term is a total derivative, and therefore it does not affect the equations of motion.

The Polyakov action is invariant under local world-sheet reparametrizations

$$\delta h^{ab} = \chi^c \partial_c h^{ab} - (\partial_c \chi^a) h^{cb} - (\partial_c \chi^b) h^{ca}, \quad \delta x^\mu = \chi^a \partial_a x^\mu,$$

We have also invariance under local Weyl transformations

$$\delta x^\mu = 0, \quad \delta h_{ab} = e^{\rho(\xi)} h_{ab}$$

In flat space-time we have ivariance under global Poincare transformations

$$\delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu}, \quad \delta h_{ab} = 0, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

Let us consider the equations of motion with respect to $h_{\mu\nu}$. Recall that for any matrix h, we have

$$\delta h^{-1} = -h^{-1} \cdot \delta h \cdot h^{-1}, \quad since \quad h \cdot h^{-1} = 1$$

Also

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\delta(h_{ab})h^{ab}$$

Using these equations, we get the equation of motion with respect to h_{ab}

$$T_{ab} = (\partial_a x^{\mu})(\partial_b x^{\nu})\eta_{\mu\nu} - \frac{1}{2}h_{ab}(\partial_c x^{\mu})(\partial_d x^{\nu})h^{cd}\eta_{\mu\nu} = 0 \tag{4}$$

This equation can be written as $T_{ab} = 0$, where T_{ab} is the Energy-Momentum tensor.

3. Nambu-Goto action

The Nambu-Goto action for the bosonic string has the form

$$S = -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{det((\partial_a x^{\mu})(\partial_b x^{\nu})\eta_{\mu\nu})} \tag{5}$$

This action can be written as

$$S = -\frac{1}{2\pi\alpha'}\int dA, \quad where \quad dA = \sqrt{\gamma_{ab}}$$

and

$$\gamma_{ab} = (\partial_a x^{\mu})(\partial_b x^{\nu})\eta_{\mu\nu}$$

is an induced metric on a two dimensional surface. In other words we embedded a two–dimensional surface into a $d-$ dimensional surface. The expression dA is an infinitesimal element of the two- dimensional surface.

IN GENERAL: We considered a zero -form (a particle), one- form (a string). We can consider an $n-$ form. An action will be proportional to the corresponding infinitesimal $n+1$ dimensional volume.

4. Equations of Motion

Before considering the equations of motion, let us note that since Polyakov action is reparametrization invariant

$$(\tau, \sigma) \to (\tau(\tau', \sigma'), \sigma(\tau', \sigma'))$$

and we have also symmetry under Weyl rescalings, we can choose the gauge, where the two dimensional metric has the form

$$h_{ab} = e^{\phi(\tau,\sigma)}\eta_{ab} = e^{\phi(\tau,\sigma)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This form of the metric is achieved using the reparametrization invariance. This gauge is called a conformal gauge.

If we insert the metric in the conformal gauge into the Polyakov action, then the conformal factor will disappear and we shall obtain

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi (\partial^a x^\mu)(\partial_a x_\mu) \tag{6}$$

Therefore, the action does not depend on $\phi(\tau, \sigma)$. However, whenever one fixes the gauge in the action (before the variation) one should be careful, since one can loose some equations of motion. For example in Electrodynamics the component A_0 of the vector potential A_μ is non-physical. But the variation of the action with respect to this component $\frac{\delta S}{\delta A_0}$ gives the Gauss law, which is physical.

Similarly, here we can choose the conformal gauge, but we should remember the corresponding equations of motion $T_{ab} = 0$,. This equation is called Virasoro constraint.

An important note: We still have a left over invariance under transformations which keep the world-sheet metric flat up to an overall factor. This is a conformal invariance, which is an infinite dimensional symmetry in the case of two dimensions (world-sheet).

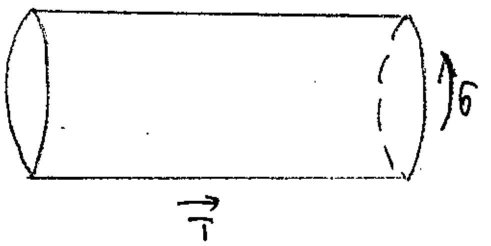
The equations of motion with respect to x^μ are

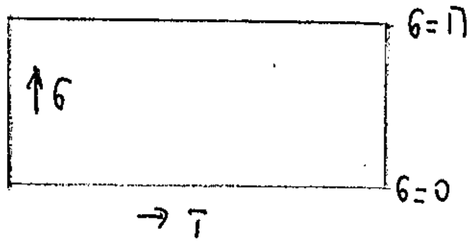
$$\Box x^\mu = (\partial_\tau^2 - \partial_\sigma^2) x^\mu = 0 \tag{7}$$

and also when varying the action we get a boundary term of the form

$$\int d\tau \left[\frac{\partial x_\mu}{\partial \sigma} \delta x^\mu |_{\sigma=0}^{\sigma=\pi} \right]$$

Besides, we can have either Open or a Closed String. Let us consider a Closed String. Boundary conditions are imposed on its endpoints, which in this case means periodicity conditions.





Here the Closed String propagates from left to right. The motion can be considered as a rectangle , where the boundary points ($\sigma = 0$ and $\sigma = \pi$) are identified. Finally,

$$x^\mu(\tau, 0) = x^\mu(\tau, \pi)$$

$$\frac{\partial x^\mu(\tau, 0)}{\partial \sigma} = \frac{\partial x^\mu(\tau, \pi)}{\partial \sigma}$$

The solution is

$$x^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}\left(\frac{\alpha_n^\mu}{n}e^{2in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n}e^{-2in(\tau+\sigma)}\right) \tag{8}$$

Let us note, that the equation of motion (7) can be written in terms of the light-cone coordinates

$$\xi^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{\partial}{\partial \xi^\pm}$$

The nonzero components of the world-sheet metric are

$$h_{+-} = h_{-+} = \frac{1}{2}, \quad h^{+-} = h^{-+} = 2.$$

The equations of motion (7) now are

$$\partial_+\partial_- x^\mu(\xi^+,\xi^-) = 0$$

They have a general solution

$$x^\mu(\xi^+,\xi^-) = x_L^\mu(\xi^+) + x_R^\mu(\xi^-)$$

The exact form for $x_L^\mu(\xi^+)$ and $x_R^\mu(\xi^+)$ depends on the boundary conditions. For example for the Closed String we have

$$x_L^\mu(\xi^+) = \frac{1}{2}x_0^\mu + \alpha' p_0^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}\frac{\tilde{\alpha}_n^\mu}{n}e^{-2in\xi^+} \tag{9}$$

$$x_R^\mu(\xi^-) = \frac{1}{2}x_0^\mu + \alpha' p_0^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}^{\infty}\frac{\alpha_n^\mu}{n}e^{-2in\xi^-} \tag{10}$$

The coordinate $x^\mu(\xi^+,\xi^-)$ must be real. Therefore, the constants x_0^μ and p_0^μ are real and

$$(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu, \quad (\alpha_n^\mu)^* = \alpha_{-n}^\mu$$

The constants x_0^μ and p_0^μ are coordinates of the center of mass of the string and its momentum. Indeed we have

$$S = \frac{1}{4\pi\alpha'}\int d^2\xi\,((\partial_\tau x^\mu)(\partial_\tau x_\mu) - (\partial_\sigma x^\mu)(\partial_\sigma x_\mu))$$

The momentum density is

$$P^\mu = \frac{\partial L}{\partial_\tau x_\mu} = \frac{1}{2\pi\alpha'}\partial_\tau x^\mu$$

Using (8) we can find the total momentum

$$p^\mu = \int_0^\pi d\sigma\, P^\mu = p_0^\mu$$

The constants x_0^μ and p_0^μ are called zero modes of the string. Obviously, except of the zero modes all other terms have zero average.

5. Open String

There are two options for the endpoints of the Open String.

- Neumann boundary conditions

$$\frac{\partial x^\mu(\tau,\sigma)}{\partial\sigma} = 0, \quad at \quad \sigma = 0, \pi$$

  These boundary conditions mean that endpoints of the Open String can be anywhere in the space.

- Dirichlet boundary conditions

$$\frac{\partial x^\mu(\tau,\sigma)}{\partial\tau} = 0, \quad at \quad \sigma = 0, \pi$$

  These boundary conditions mean that the position of endpoints is fixed in space. We shall consider this situation in more details later.

  The solution will be

$$x^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\alpha_n^\mu}{n} e^{-in\tau}\cos(n\sigma)$$

  this solution satisfies Neumann boundary conditions.