LECTURE 1. INTRODUCTION.

1. Dual Models

In the 1960s physicists were trying to understand experimental data, obtained from studies of strong interactions. There were observed many particles (resonances) with growing spins. It was necessary to bring these data into some kind of system.

The observations showed that masses of the resonances where were linearly growing with spins i.e., there was a relation

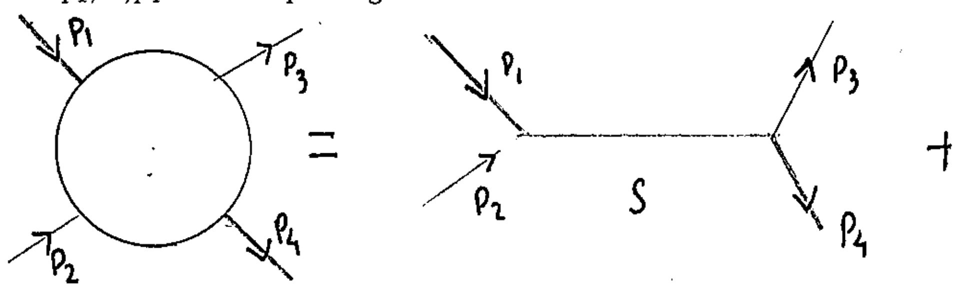
$$m^2 = \frac{J}{\alpha'} + \alpha_0$$

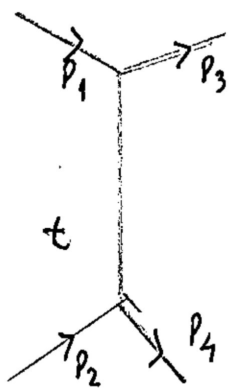
with $\alpha' \sim 1 GeV^{-2}$. This relation has been checked up to spin $J = \frac{11}{2}$.

1.1. s-t duality. Let us consider a scattering of two hadrons on two hadrons. This scattering is described by Mandelstam Variables

$$s = -(p_1 + p_2)^2, \quad t = -(-p_1 + p_3)^2, \quad u = -(-p_1 + p_4)^2,$$

Here $p_1, \ldots, p_4$ are corresponding four-momenta.





We have in the mostly "+" signature

$$p_1 + p_2 = p_3 + p_4, \quad and \quad p_i^2 = -m_i^2$$

Therefore

$$s = m_1^2 + m_2^2 - 2p_1p_2, \quad t = m_1^2 + m_3^2 + 2p_1p_3, \quad u = m_1^2 + m_4^2 + 2p_1p_4$$

Adding them up

$$s + t + u = 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2p_1(-p_3 - p_4 + p_2)$$

We finally get

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

Let us assume that the scattering particles belong to an adjoint representation of the flavour group. That means that the quantum numbers of ith meson are described by the corresponding matrix λ_i. The entire amplitude will have a factor $tr(\lambda_1\lambda_2\lambda_3\lambda_4)$. But since $tr()$ is invariant under cyclic permutations of the matrices, we should have the same symmetry under permutation of the four momenta as well. This in turn means the symmetry between s and t.

$$-(p_1+p_2)^2 \to -(p_1-p_3)^2 = t$$

$$-(-p_1+p_3)^2 \to -(p_3+p_4)^2 = -(p_1+p_2)^2 = s$$

In other words the amplitude should satisfy

$$A(s,t) = A(t,s)$$

Suppose that the external particles are scalars and internal particle (the one that is exchanged by the external particles) has mass M and spin J. Then at high energies the amplitude in the t- channel will have the form:

$$A_J(s,t) \sim \frac{g^2(-s)^J}{t-M^2}$$

This amplitude becomes more and more divergent for higher J. If we consider loop diagrams, then we shall have the amplitude in four dimensions

$$\int d^4p \frac{A^2}{p^4}$$

This expression is divergent if $J > 1$. On the other side, if one has an infinite number of particles with different masses and spins then in the $t-$ channel we have

$$A_J(s,t) \sim \sum \frac{g_J^2(-s)^J}{t-M_J^2}$$

where g_J are coupling constants and M_J are masses. They can depend on the spins. Then, in principle, the amplitude can turn out to be finite. There is another argument why the number of the particles must be infinite: the amplitude $A_J(s,t)$ does not have poles in the s-channel. Indeed, for the fixed t the amplitude is an *entire* function of s. On the other hand, if the sum is infinite, but each term is finite then the sum can be divergent for a finite value of s. These values of s will become poles in the s-channel.

If this duality indeed takes place, the situation is different from the one in the usual Quantum Field Theory. In the Quantum Field Theory we have to compute the amplitudes in both s- and t- channels and sum them up. Here it is enough to consider only one channel, either s or t.

In 1968 G.Veneziano postulated an amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

where $\Gamma(u)$ is the Gamma function

$$\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dt, \quad for \quad Re(u) > 0.$$

and $\alpha(s) = \alpha(0) + \alpha' s$. Let us introduce also the Beta-function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Then can write the Veneziano amplitude also in the form

$$A(s,t) = B(-\alpha(s), -\alpha(t))$$

Recall, that if u is a positive integer, then

$$\Gamma(u) = (u-1)!, \quad \Gamma(1) = 1$$

We have also

$$\Gamma(u) = \frac{\Gamma(u+1)}{u}$$

This gives us a definition of the Gamma function for $Re(u) > -1$, since the right hand side of the equation above is well defined in this area. Moreover, we can see, that $\Gamma(u)$ has a simple pole in $u = 0$ and the residue is equal to 1.

Using n iterations we get

$$\Gamma(u) = \frac{\Gamma(u+n)}{u(u+1)...(u+n-1)}$$

for any positive n. In the area $Re(u) > -n$ the function $\Gamma(u+n)$ has a unique integral representation. Therefore we managed to analytically continue the Gamma function to this area. But since n is arbitrary, we can analytically continue the Gamma function to the entire complex plane.

We have poles at $0, -1, ..., u-n+1$. Near the poles

$$\Gamma(u) \sim \frac{1}{u+n}\frac{(-1)^n}{n!}$$

Therefore we have

$$\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \sim \frac{(-1)^n}{n!}\frac{1}{s+n}\frac{\Gamma(t)}{\Gamma(t-n)}$$

and

$$\frac{\Gamma(t)}{\Gamma(t-n)} = (t-1)...(t-n+1)(t-n)$$

From these relations we get for the Veneziano amplitude

$$A(s,t) = -\sum_{n=0}^{\infty}\left(\frac{(\alpha(t)+1)(\alpha(t)+2)...(\alpha(t)+n)}{n!}\frac{1}{\alpha(s)-n}\right)$$

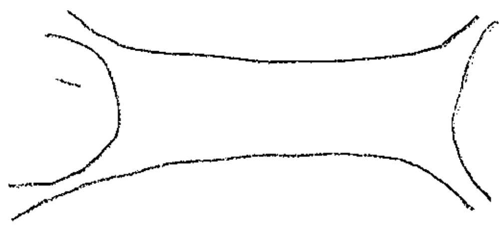
or, equivalnetly

$$A(s,t) = -\sum_{n=0}^{\infty}\left(\frac{(\alpha(s)+1)(\alpha(s)+2)...(\alpha(s)+n)}{n!}\frac{1}{\alpha(t)-n}\right)$$

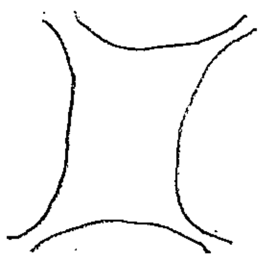
In this equation the poles correspond to exchanges of the intermediate particles with masses

$$M^2 = \frac{(n-\alpha(0))}{\alpha'}$$

and with higher spins. These amplitudes can be obtained from theory of strings (Y.Nambu, T.Goto, H-B. Nielsen, L.Susskind). Let us consider interactions between open strings



Apparently, this diagram can be continuously deformed to the diagram



and this procedure makes the duality between s- and t- channels apparent.
However the theory of strings seemed problematic because

- The spectrum of the strings contained particles with negative mass squared (so called tachyons).
- Some theories of strings contained a massless particle with spin 2.
- It was not clear how to introduce fermions

Moreover experimental data from SLAC was suggesting that at high energies hadrons behaved like point particles, not like strings. Then the Quantum Chromodynamics (QCD) was developed - an area of the Quantum Field Theory which today is commonly accepted as a language describing the Strong Interactions.

2. Fixed - angle scattering

Let us consider the case when

$$m_1 = m_2 = m_3 = m_4 \equiv m$$

In the center of mass system the four momenta have the form

$$p_1 = \left(\frac{\sqrt{s}}{2}, \vec{k}\right), \quad p_2 = \left(\frac{\sqrt{s}}{2}, -\vec{k}\right), \quad p_3 = \left(\frac{\sqrt{s}}{2}, \vec{l}\right), \quad p_4 = \left(\frac{\sqrt{s}}{2}, -\vec{l}\right)$$

The scattering angle is the angle between the vectors $\vec{k}$ and $\vec{l}$. Let us consider the limit $s/m^2 \to \infty$ with s/t and s/u being fixed. There is also a Regge limit. In this limit we have $s/m^2 \to \infty$ with t being fixed which, means that the scattering angle $\theta \to 0$.

We have

$$\begin{aligned} t &= -(\vec{k} - \vec{l})^2 = -(\vec{k}^2 + \vec{l}^2 - 2\vec{k}\cdot\vec{l}) = -\left(\frac{s}{4} - m^2 + \frac{s}{4} - m^2 - 2|\vec{k}||\vec{l}|\cos\theta\right) = \\ &= -(s - 4m^2)\sin^2\frac{\theta}{2} \end{aligned}$$

since

$$-\frac{s}{4} + \vec{k}^2 = -\frac{s}{4} + \vec{l}^2 = -m^2$$

Because we take large s, we have

$$t \sim -s \cdot \sin^2\frac{\theta}{2}$$

Let us recall that in the limit $x \to \infty$

$$\Gamma(x) \sim x^{x-1/2} e^{-x}\sqrt{2\pi}$$

Then we have

$$\left|\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(-\alpha' s - \alpha' t)}\right| \sim \frac{|\alpha' s|^{-\alpha' s}|\alpha' t|^{-\alpha' t}}{|\alpha' s + \alpha' t|^{-\alpha' s - \alpha' t}}.$$

Therefore, when $t \sim -s \cdot \sin^2\frac{\theta}{2}$ we have

$$B(s,t) = \frac{|\sin^2\frac{\theta}{2}|^{\alpha' s\cdot\sin^2\frac{\theta}{2}}}{|\cos^2\frac{\theta}{2}|^{-\alpha' s\cdot\cos^2\frac{\theta}{2}}}$$

Let us introduce a new variable $x = \sin^2\frac{\theta}{2}$. Then we have approximately

$$B(s,t) \sim \exp(\alpha' s \cdot x\ln x + \alpha' s \cdot (1-x)\ln(1-x))$$

In our case $0 \le x \le 1$. As one can see, the amplitude falls exponentially in the minimum when $x = 1/2$, since we have

$$\exp(-\alpha' s \ln 2)$$

Which is not correct for tree level scattering for hadrons.