Duality-symmetric formulation of electrodynamics and (chiral) $p-$form generalizations

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References

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Standard approach to field theories

The Lorentz-covariant field variable is taken in the same representation as that of the little group carried by the corresponding particle

Some well-known examples

- Trivial representation of the little group corresponds to the spin-zero particle. Lorentz covariant variable – scalar field.
- Vector representation of the little group corresponds to the spin-one particle and is described by a Lorentz vector field (Maxwell potential).
- Symmetric tensor of the little group corresponds to the spin-two particle and is described by symmetric Lorentz tensor (metric) satisfying linearised Einstein equations (Fierz-Pauli).

Particles and fields

Wigner classification of particles $\leftrightarrow$ field equations (unique?)

For massless spin-zero particle the simplest option is the Klein-Gordon equation

$$\Box\,\phi = 0$$

The scalar here is a single field that carries one degree of freedom: trivial representation of the massless little group. The Lagrangian is

$$\mathcal{L} \sim \frac{1}{2}\phi\,\Box\,\phi$$

Alternative

An alternative formulation of the scalar field is given by so-called Notoph Lagrangian by Ogievetsky and Polubarinov (1966):

$$\mathcal{L} \sim \partial^{\mu} B_{\mu\nu} \partial_{\lambda} B^{\lambda\nu}$$

Notoph explained

The scalar Notoph Lagrangian

$$\mathcal{L} \sim \partial^\mu B_{\mu\nu} \partial_\lambda B^{\lambda\nu}$$

can be written in a more conventional form using different variables: $C_{\mu_1 \dots \mu_{d-2}} = \epsilon_{\mu_1 \dots \mu_d} B^{\mu_{d-1}\mu_d}$. Then, the Lagrangian is a regular Maxwell-type Lagrangian for the $(d-2)-$form field

$$\mathcal{L} \sim (\partial_{[\mu_1} C_{\mu_2 \dots \mu_{d-1}]})^2$$

which describes a $(d-2)-$form representation of the little group, dual to scalar.

Interactions depend on the formulation of the free theory

Interacting spin-zero particles

The scalar-field formulation allows for straightforward generalisation to non-linear theory with arbitrary potential:

$$\mathcal{L} \sim \frac{1}{2}\phi\,\Box\,\phi + V(\phi)\,.$$

Instead, the notoph formulation does not allow for any non-derivative self-interactions (those would spoil the gauge symmetry)!

Moral of the story

The choice of the free field formulation plays an important role in deriving possible interacting theories.
Therefore, before addressing the problem of the interacting p-forms, we should find a convenient action for the free fields.

Duality symmetry of Maxwell equations

The most familiar example of duality symmetry – free Maxwell eq.'s:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \quad \vec{\nabla} \cdot \vec{E} = 0\,,$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}\,, \quad \vec{\nabla} \cdot \vec{B} = 0\,,$$

invariant with respect to the duality rotations:

$$\vec{E} \to \cos\alpha\,\vec{E} + \sin\alpha\,\vec{B}\,,$$

$$\vec{B} \to -\sin\alpha\,\vec{E} + \cos\alpha\,\vec{B}\,.$$

Discreet duality – exchange of the electric $\vec{E}$ and magnetic $\vec{B}$ fields:

$$\vec{E} \to \vec{B}\,, \quad \vec{B} \to -\vec{E}\,.$$

Duality symmetry of Maxwell equations

When the electromagnetic field is coupled to charged matter,

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j_e}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 4\pi \rho_e\,,$$

the duality symmetry is broken, unless one introduces magnetic charges – monopoles. These form a magnetic current $\overrightarrow{j_m}$:

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{j_m}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 4\pi \rho_m\,.$$

The Maxwell equations remain duality invariant if the duality rotates also the four-vector currents $j_e^\mu = (\rho_e, \overrightarrow{j_e})$, $j_m^\mu = (\rho_m, \overrightarrow{j_m})$:

$$j_e^\mu \to \cos\alpha\, j_e^\mu + \sin\alpha\, j_m^\mu\,,$$

$$j_m^\mu \to -\sin\alpha\, j_e^\mu + \cos\alpha\, j_m^\mu\,.$$

Duality symmetry of electromagnetic equations

Maxwell action (with one potential) is not duality symmetric:

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x(\overrightarrow{E}^2 - \overrightarrow{B}^2)\,.$$

It changes the sign under discreet duality transformations.

Democracy requires employing two vector potentials: A^1_μ and A^2_μ with field strengths $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ $(a = 1, 2)$. Free Maxwell equations are equivalent to (twisted self-) duality relation:

$$F^a_{\mu\nu} = \epsilon^{ab} \star F^b_{\mu\nu}\,,$$

where

$$\star F^b_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho}F^{b\,\lambda\rho}\,, \quad \epsilon^{ab} = -\epsilon^{ba}\,, \quad \epsilon^{12} = 1$$

Duality

A $p-$form and its dual

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F \,, \quad F = dA \,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly particles of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, dual to each other.

Attention!

Dual formulations do not admit the same interacting deformations!

Duality-symmetric equations

Maxwell action for $p-$forms and $(d-2-p)-$forms describes the same particle content.
When $d = 2p + 2$, the dual variables are of the same type and the Maxwell action itself takes the same form in both variables.

Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$F = \pm \star G\,, \quad F = dA\,, \quad G = dB$$

Duality-symmetric formulations

Zwanziger '70,..., Gaillard-Zumino '80, Bialynicki-Birula '83,..., Schwarz-Sen '93, Gibbons-Rasheed '95, Pasti-Sorokin-Tonin '96, Cederwall-Westerberg '97, Rocek-Tseytlin '99, Kuzenko-Theisen '00, Ivanov-Zupnik '02,...

Chiral $p-$forms in $d = 4k + 2$ Minkowski space

Minkowski vs Euclidean

Since $\star^2 = (-1)^{\sigma+p+1}$ where σ is the number of time directions, only even-forms can be self-dual (chiral) in Minkowski space.

$p = 2k$ forms in $d = 4k + 2$ dimensions

For even $p-$form potentials in special dimensions the corresponding particles are not irreducible but contain two irreps — chiral and anti-chiral halves.

Self-dual (Chiral) fields

There are special representations of the Poincaré algebra which are described by self-dual forms. The covariant equations describing such representations are given as:

$$F = \pm \star F\,, \quad F = dA$$

which implies the regular “Maxwell equations” $d \star F = 0$.

Lagrangian?

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91, ..., Pasti-Sorokin-Tonin '95,..., Sen '15,...

Pasti-Sorokin-Tonin formulation

There are many different formulations for free chiral p-forms: the most economic covariant one is that of Pasti, Sorokin and Tonin. E.g., PST action for chiral two-form in six dimensions:

$$S = -\int d^6x \Big[\frac{1}{6} F_{\mu\nu\lambda} F^{\mu\nu\lambda} + \frac{1}{2(\partial a)^2} \partial^\lambda a \mathcal{F}_{\lambda\mu\nu} \mathcal{F}^{\mu\nu\rho} \partial_\rho a \Big] \,,$$

where

$$F_{\mu\nu\lambda} = 3\, \partial_{[\mu} \varphi_{\nu\lambda]} \,, \quad \mathcal{F}_{\mu\nu\lambda} = F_{\mu\nu\lambda} - \frac{1}{6} \varepsilon_{\mu\nu\lambda\alpha\beta\gamma} F^{\alpha\beta\gamma} \,,$$

The field a is called "PST scalar", is an auxiliary field that has to satisfy the condition: $\partial_\mu a \, \partial^\mu a \neq 0 \,.$

New action for Chiral fields

Lagrangian

$$\mathcal{L} = (F + a\,Q)^2 + 2\,a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

Symmetries

$$\begin{aligned} \delta A = dU\,; \qquad & \qquad \delta R = dV\,; \\ \delta A = -\,a\,da \wedge W\,, \quad & \delta R = da \wedge W\,; \\ \delta A = -\,\frac{a\,\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,, & \\ \delta a = \varphi\,, \quad \delta R = \frac{\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,. & \end{aligned}$$

Equations and symmetries

Equations

$$E_a \equiv \frac{\delta \mathcal{L}}{\delta a} \equiv (F + a\, Q) \wedge \star Q + F \wedge Q = 0\,,$$
$$E_A \equiv \frac{\delta \mathcal{L}}{\delta A} \equiv d\,[\star(F + a\, Q)] + da \wedge Q = 0\,,$$
$$E_R \equiv \frac{\delta \mathcal{L}}{\delta R} \equiv d\,[a \star (F + a\, Q)] - da \wedge F = 0\,.$$

Relations

$$E_R - a\, E_A = da \wedge [F + a\, Q - \star(F + a\, Q)] = 0$$

From here (for $(da)^2 \neq 0$):

$$F + a\, Q - \star(F + a\, Q) = 0$$

and $E_a \equiv [F + a\, Q - \star(F + a\, Q)] \wedge Q = 0$ follows from $E_A = 0 = E_R$.

The spectrum

Consequences of e.o.m.

Equations imply:

$$da \wedge dR = 0$$

which implies that R is pure gauge. In the $R = 0$ gauge, we get:

$$F = \star F$$

Thus the propagating d.o.f. consist of a single self-dual $p-$form.

A Generalisation

An interesting generalisation of the Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\, f(a)\, (\sqrt{a}\, F + \frac{1}{\sqrt{a}}\, Q)^2 + \, f(a)\, F \wedge \, Q\,.$$

For $f(a) \sim 1/a$, this Lagrangian is equivalent to the chiral one written earlier and describes a single chiral p-form carried in field φ.

For $f(a) \sim a$, it describes an anti-chiral $p-$form field carried by R.

$d = 2$

Another parametrization gives:

$$\mathcal{L}_{\pm} = -\frac{1}{8}\left[(r+1)\partial_\mu\varphi \pm (r-1)\partial_\mu\tilde{\varphi}\right]^2 + \frac{1}{4}\,\epsilon^{\mu\nu}\,r\,\partial_\mu\varphi\,\partial_\nu\tilde{\varphi}\,,.$$

where different signs correspond to different chiralities. Here $\varphi = \varphi_+ + \varphi_-$ and $\tilde{\varphi} = \varphi_+ - \varphi_-$. The two actions transform into each other under $\varphi \leftrightarrow \tilde{\varphi}\,,\ r \to -r$.

A compact form

The Lagrangian in a simpler form

$$\mathcal{L} \sim -\mathcal{M}_{IJ}\, F^I \wedge \star F^J - \mathcal{K}_{IJ}\, F^I \wedge F^J \,,$$

with

$$\mathcal{M}_{IJ} = \begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}, \quad \mathcal{K}_{IJ} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \quad F^I = \begin{bmatrix} F \\ Q \end{bmatrix},$$

where F^I is a two-vector with $p+1$-form components. $\mathcal{M}$ is of rank one. The "background matrix" $\mathcal{E} = \mathcal{M} + \mathcal{K}$ is invertible.

An observation

The same action with the inverted background matrix $\mathcal{E}^{-1}$ describes the same degrees of freedom, exchanging the roles of A and R.

Democratic formulation for $p-$form in d dimensions

Lagrangian

$$\mathcal{L} = (F + a\,P)^2 + (G + a\,Q)^2 - 2\,a\,Q \wedge F + 2\,a\,G \wedge P$$

where $F = dA\,,\ G = dB\,,\ P = dS\,,\ Q = dR$. The fields A and S are $p-$forms, while B and R are $(d-p-2)-$forms.

This is a democratic formulation for $p-$form fields (together with dual $(d-p-2)-$form field) in d dimensions. The equations imply that S, R are pure gauge (as is the field a), and the only physical d.o.f. are in A, B, satisfying the duality relation:

$$dA = \star dB$$

Duality-symmetric Electromagnetism

The Lagrangian for a single massless spin-one field

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} H^b_{\mu\nu} H^{b\mu\nu} + \frac{a(x)}{4}\, \epsilon_{bc}\, \varepsilon^{\mu\nu\lambda\rho}\, F^b_{\mu\nu}\, Q^c_{\lambda\rho}$$

where $H^b_{\mu\nu} \equiv F^b_{\mu\nu} + a\, Q^b_{\mu\nu}$, $b = 1, 2$, and

$$F^b_{\mu\nu} = \partial_\mu\, A^b_\nu - \partial_\nu\, A^b_\mu\,, \quad Q^b_{\mu\nu} = \partial_\mu\, R^b_\nu - \partial_\nu\, R^b_\mu\,.$$

This Lagrangian describes a single Maxwell field, using 4 vectors and 1 scalar. Any solution of the e.o.m. is gauge equivalent to that of

$$R^b_\mu = 0, \qquad \star F^a_{\mu\nu} + \epsilon^{ab}\, F^b_{\mu\nu} = 0\,,$$

with a single propagating Maxwell field.

Non-linear electrodynamics: Ansatz

Ansatz for the consistent non-linear Lagrangian

$$\mathcal{L} = a\,\epsilon_{bc} F^b \wedge Q^c + f(U^{ab}, V^{ab})$$

where

$$U^{ab} \equiv \frac{1}{2}\,H^a_{\mu\nu}\,H^{b\mu\nu}\,, \quad V^{ab} \equiv \frac{1}{2}\,H^a_{\mu\nu}\,\star H^{b\mu\nu}$$

All symmetries are built in, except for the shift of a. The latter will fix the form of $f(U, V)$.

Imposing the missing symmetry

Shift symmetry $\delta a = \varphi$

Equations of motion for a:

$$E_a \equiv Q^b \wedge K_b = 0 \,,$$

where

$$K_a \equiv (f^U_{ab} + f^U_{ba}) \star H^b - (f^V_{ab} + f^V_{ba})\, H^b - \epsilon_{ab}\, H^b \,,$$

and $f^U_{ab} \equiv \partial f / \partial U_{ab}$, $f^V_{ab} \equiv \partial f / \partial V_{ab}$ ($f^U_{21} \equiv 0 \equiv f^V_{21}$).

Note, that $E_{R^b} - a\, E_{A^b} = da \wedge K_b = 0$, which implies $K_b = 0$ when

$$K_a \pm \epsilon_{ab} \star K_b \equiv 0$$

Then, the $E_a = 0$ is redundant, which means that the shift symmetry for a is present.

The general democratic non-linear electrodynamics

Solution

The equation $K_a \pm \epsilon_{ab} \star K_b \equiv 0$ implies

$$\pm\, \delta^{ac}\, (f^U_{cb} + f^U_{bc}) - \epsilon^{ac}\, (f^V_{cb} + f^V_{bc}) + \delta^a_b = 0$$

The general solution gives the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\, G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\tfrac{1}{2}\, G_{\mu\nu}\, G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

Reminder: non-linear electrodynamics in the conventional language

$$S = \int \mathcal{L}(s, p)\, d^4x, \quad s \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad p \equiv \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu}$$

Duality symmetry

Discreet duality symmetry

Under the discrete duality,

$$\lambda_1 \to -\lambda_1\,, \qquad \lambda_2 \to -\lambda_2$$

Theories with such symmetry will satisfy:

$$g(-\lambda_1, -\lambda_2) = g(\lambda_1, \lambda_2)$$

Duality symmetry

Continuous duality symmetry

Under continuous duality symmetry,

$$\lambda_1 \to \cos(2\alpha)\,\lambda_1 + \sin(2\alpha)\,\lambda_2\,,$$
$$\lambda_2 \to -\sin(2\alpha)\,\lambda_1 + \cos(2\alpha)\,\lambda_2$$

Theories with such symmetry will have:

$$g(\lambda_1, \lambda_2) = h(w)\,, \qquad w = \sqrt{\lambda_1^2 + \lambda_2^2}$$

The corresponding Lagrangian is given as:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,,$$

where w can be also given as:

$$w = \sqrt{-\det \mathcal{H}}\,, \qquad \mathcal{H}^{ab} \equiv (\star H^a_{\mu\nu} - \epsilon^{ac} H^c_{\mu\nu})(\star H^{b\mu\nu} - \epsilon^{bd} H^{d\mu\nu})/2$$

Conformal symmetry

Requirement of conformal symmetry

Requirement of conformal invariance translates into:

$$\lambda_1 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1} + \lambda_2 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2} = g(\lambda_1, \lambda_2)$$

which can be solved, e.g. as:

$$g = \lambda_1 \, \tilde{g}(\lambda_1 / \lambda_2)$$

Conformal and duality-symmetric theory

Conformal symmetry for duality-symmetric theories

This case gives:

$$w\,\frac{\partial h(w)}{\partial w} = h(w)\,,$$

which is solved by a linear function:

$$h(w) = \delta\, w$$

General conformal and duality-symmetric electrodynamics is given by the one-parameter Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\,H^b \wedge \star H^b + a\,\epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

Equations of motion

Equations

E.o.m. imply in $R^a = 0$ gauge:

$$\star F^1 + F^2 = g_2 \left(\star F^1 - F^2\right) - g_1 \star \left(\star F^1 - F^2\right),$$

where $g_1 \equiv \partial g/\partial\lambda_1,\ g_2 \equiv \partial g/\partial\lambda_2$.
One can solve from here F^1 in terms of F^2:

$$F^1 = \alpha(s,p)F^2 + \beta(s,p)\sigma F^2\,,$$

where $s = \frac{1}{2}F^2_{\mu\nu}F^{2\mu\nu}\,,\ p = \frac{1}{2}F^2_{\mu\nu}\star F^{2\mu\nu}$. One can now make contact with the single-field formalism with Lagrangian $\mathcal{L}(s,p)$ via

$$\alpha(s,p) = -\frac{\partial\mathcal{L}}{\partial p}\,, \qquad \beta(s,p) = \frac{\partial\mathcal{L}}{\partial s}$$

Map between different formulations

The relation between single and double potential formulations

The relation between derivatives of Lagrangians in both formulations:

$$g_1 = \frac{2\,\alpha}{\alpha^2 + (\beta+1)^2}\,, \qquad g_2 = \frac{\alpha^2 + \beta^2 - 1}{\alpha^2 + (\beta+1)^2}\,,$$

where g is a function of λ_1, λ_2, which can also be expressed in terms of α, β, s, p:

$$\lambda_1 = 2\,\alpha\,(1+\beta)\,s - [\alpha^2 - (1+\beta)^2]\,p\,,$$
$$\lambda_2 = [\alpha^2 - (1+\beta)^2]\,s + 2\,\alpha\,(1+\beta)\,p\,,$$

while w is given as:

$$w \equiv \sqrt{\lambda_1^2 + \lambda_2^2} = (\alpha^2 + (\beta+1)^2)\,\sqrt{s^2 + p^2}$$

Map for duality-symmetric theories

The $SO(2)$ invariant case

The relation between the two formulations is given in this case by:

$$\frac{\lambda_1}{w}h' = \frac{2\,\alpha}{\alpha^2 + (\beta+1)^2}\,, \quad \frac{\lambda_2}{w}h' = \frac{\alpha^2+\beta^2-1}{\alpha^2+(\beta+1)^2}$$

which implies the duality-symmetry condition for the single-potential formulation

$$\beta^2 + \frac{2s}{p}\alpha\beta - \alpha^2 = 1\,,$$

and:

$$(\alpha s + (\beta+1)p)\; h'\Big|_{w=\sqrt{s^2+p^2}(\alpha^2+(\beta+1)^2)} = \alpha\sqrt{s^2+p^2}$$

Examples: ModMax and BB

The conformal duality-symmetric electrodynamics

The conformal and duality-symmetric Electrodynamics:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\, \epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

can be translated to single-potential formulation

$$L(s,p) = -\cosh\gamma\, s + \sinh\gamma \sqrt{s^2 + p^2}$$

using a parametrization: $\delta = \coth\frac{\gamma}{2}$. This is so-called ModMax theory. In the special case of $\delta = 1$, the map breaks down. There, the single-field formulation does not exist. This corresponds to Bialynicki-Birula Electrodynamics.

Example: Generalized Born-Infeld theory

Generalized Born-Infeld theory

The conventional Lagrangian (T, γ are constants):

$$L_{GBI} = \sqrt{UV} - T, \quad U \equiv 2u + e^{\gamma}\, T, \quad V \equiv -2v + e^{-\gamma}\, T\,,$$

where $u \equiv (s + \sqrt{p^2 + s^2})/2$, $v \equiv (-s + \sqrt{p^2 + s^2})/2$.

Democratic formulation

The duality-symmetric Lagrangian is $\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)$, where in this case $h(w)$ is implicitly given by:

$$h(\lambda) = 4\,T\,\sinh^2\frac{\lambda}{2}\,\cosh(\lambda+\gamma)\,,$$

$$w(\lambda) = -4\,T\,\cosh^2\frac{\lambda}{2}\,\sinh(\lambda+\gamma)\,.$$

Summary

Main statements

- The choice of the free Lagrangian matters.
- The Lagrangian can manifest duality symmetry, and in general, democracy between electric and magnetic degrees of freedom without compromising manifest Poincaré symmetry.
- Self-interactions are simple and tractable in democratic formulation.

Outlook

A list of related ambitious problems

- Duality-symmetric formulation for non-abelian gauge theory.
- Interacting theory of non-abelian (chiral) $p-$forms.
- Extensions to Gravity and beyond.

Thank you for your attention!