On amplitudes and superamplitudes of 10D SYM and 11D supergravity.

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based on PRL 118(2017)3, JHEP 11(2018)017, 05(2018)103, JHEP 11(2019)087 and current study

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INTERLUDE

- In recent 2312.15111[hep-th] and 2312.12592[hep-th] Herderschee and Maldacena calculated M-theory amplitudes from Matrix theory (BFSS model) to show that this works in wider range of relevant parameters then expected.
- They had to compare the results with 11D SUGRA amplitudes,
- but, not having in hand the 11D covariant super-amplitude formalism, they restricted their 'field theory' calculations by the case where all scattered particles are in the same 4-plane of 11D spacetime so that 4D spinor helicity and superamplitude formalism can be used.
- This restricted the generality of their arguments and corresponding conclusions for higher point amplitudes.

- However, the manifestly Lorentz covariant 11D spinor helicity, amplitude and superamplitude formalisms do exist [I.B. PRL2017, JHEP2018,18,19]
- and probably the above problem suggests to come back to its further development and applications.
- An additional reason for this gives e.g. recent 2402.03453 [hep-th] by Renata Kallosh.

Outline

- Great progress in amplitude calculations, including multiloop amplitudes, reviewed in [Bern, Carrasco, Dixon, Johansson and Roiban, Fortsch.Phys. 2011], [Benincasa, Int.J.Mod.Phys. A2014],

  [Evlang and Huang, "Scattering amplitudes...", CUP 2015]

  is related in its signiificant part to the use of twistor-like and (super)twistor methods.
- In particular, let us refer on BCFW approach first developed for tree gluon amplitudes in [R. Britto, F. Cachazo, B. Feng and E. Witten, PRL2005] (see also [Britto, Cachazo, Feng, NPB05])
- and generalized for tree and loop *super*amplitudes of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SG in
  - Arkani-Hamed, Cachazo, Kaplan, JHEP 2010 [arXiv:0808.1446[hep-th]],
  - Brandhuber, Heslop, Travaglini, PRD 2008 [arXiv:0807.4097 [hep-th]].

- The list of important papers in this direction certainly includes
  - Bianchi, Elvang, D. Freedman, JHEP 2008 [arXiv:0805.0757 [hep-th]],
  - Drummond, Henn, Korchemsky, E. Sokatchev, NPB 2010 [arXiv:0807.1095],
  - Drummond, Henn, Plefka, JHEP 2010 [arXiv:0902.2987 [hep-th]],

  and many others... (Sorry for missed references!)

Main elements

Main elements used in the D=4 amplitude calculations are:

- spinor helicity variables (essentially four dimensional!),
- on-shell superfields,
- superamplitudes=superfield description of the amplitudes=multiparticle generalization of the on-shell superfields.

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In this talk

- In this talk I will describe their 10D and 11D cousins,
- discuss their properties,
- and indicate their origin in the spinor moving frame formulation of the superparticle models (classical and quantum).

Higher D generalizations of spinor helicity formalism and (super)amplitudes

- [Cheung and Donal O'Connell JHEP 2009] generalization to D=6.
- For D=10: [Caron-Huot+ O'Connell JHEP 10]: i) D=10 spinor helicity formalism and ii) "Clifford superfield" description of tree D=10 SYM superamplitudes (quite non minimal $\Rightarrow$ it is not easy to use it).
- The spinor helicity formalism from [Caron-Huot and O'Connell JHEP 2010] was mainly used in the context of type IIB supergravity: [Boels, O'Connell, JHEP 12, Boels PRL 12, Wang, Yin, PRD 15, R. Kallosh 2402.03453 [hep-th]].
- In this talk, based on Phys.Rev.Lett.118(2017)3, JHEP 11(2018)017, 05(2018)103, 11(2019)087 and current study, we describe the generalization of the spinor helicity formalism, as well as on-shell superfield description for D=11 SUGRA and D=10 SYM superamplitudes.
- Actually we have proposed - and are elaborating- two approaches:
  - Constrained superamplitude formalism and
  - almost unconstrained analytic superamplitude formalism.
- These both are the subject of present talk.

What was done in PRL 2017, JHEP, 2018, 2018, 2019

- In more details:
- The staring point of this work was the observation that 10D spinor helicity variables of [Caron-Huot+O'Connell 2010] can be identified with
  - spinor moving frame variables [Bandos, Zheltukhin 91-95], [Bandos, Nurmagambetov 96], ... or, equivalently, with
  - D=10 Lorentz harmonics [Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91]
    - This observation was made independently in [Uvarov CQG 2016, arXiv:1506.01881] and used their to develop 5D spinor helicity formalism.
- This allowed us
  - to find immediately the spinor helicity formalism for 11D amplitudes [2017],
  - to propose a simpler constrained superfield formalism for superamplitudes of D=10 SYM (constrained superfields versus Clifford superfields).
  - and to develop the constrained superamplitude formalism for $D = 11$ SUGRA [2017, 2018].
  - To write a candidate BCFW recurrent relations for 10D and 11D superamplitudes [2017,2018] (which are still to be understood better!).
- To find an (almost unconstrained) analytic superamplitude formalism for $D = 11$ SUGRA and 10D SYM [2018].
- To obtain polarized scattering equation for 11D SUGRA and to relate it with 11D ambitwistor superstring [2019].

Outline

Bosonic spinors and spinor helicity formalism.

- In the spinor helicity formalism for D=4 *on-shell* amplitudes

$$\mathcal{A}(1,..,n) := \mathcal{A}(p_{(1)},\varepsilon_{(1)};...;p_{(n)},\varepsilon_{(n)}) = \mathcal{A}(\lambda_{(1)},\bar{\lambda}_{(1)};\ldots;\lambda_{(n)},\bar{\lambda}_{(n)}) \; .$$

the (light-like) momenta $p_{\mu(i)}$ and polarizations of the external particles are described by the bosonic Weyl spinors $\lambda^{A}_{(i)} = (\bar{\lambda}^{\dot{A}}_{(i)})^*$. In particular,

$$p_{\mu(i)}\sigma^{\mu}_{A\dot{A}} = 2\lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)} \qquad \Leftrightarrow \qquad p_{\mu(i)} = \lambda_{(i)}\sigma_{\mu}\bar{\lambda}_{(i)}, \qquad \mu = 0,...,3$$

where $\sigma^{\mu}_{A\dot{A}}$ are relativistic Pauli matrices, $A = 1,2$, $\dot{A} = 1,2$, and

$$\sigma^{\mu}{}_{A\dot{A}}\sigma_{\mu B\dot{B}} \equiv 2\epsilon_{AB}\epsilon_{\dot{A}\dot{B}} \quad \Rightarrow \quad \boxed{p_{\mu i}p^{\mu}_{i} = 0} \; .$$

- Indeed, in the convenient notation

$$< ij > \equiv < \lambda_{(i)}\lambda_{(j)} > = \epsilon_{AB}\lambda^{A}_{(i)}\lambda^{B}_{(j)} \; , \qquad [ij] := [\bar{\lambda}_{(i)}\bar{\lambda}_{(j)}] = \epsilon_{\dot{A}\dot{B}}\bar{\lambda}^{\dot{A}}_{(i)}\bar{\lambda}^{\dot{B}}_{(j)} \; .$$

- we find that, as $\epsilon_{AB} = -\epsilon_{BA}$ and spinors are bosonic, $< ji > = - < ij >$ and $[ii] = 0$, so that $p_{i\mu}p^{\mu}_{i} = 2 < ii > \cdot [ii] \equiv 0$.

Helicity

- The amplitude should obey the *helicity constraints*,

$$\hat{h}_{(i)}\mathcal{A}(1,...,n) = h_i\mathcal{A}(1,...,n) \,, \qquad \hat{h}_{(i)} := \frac{1}{2}\bar{\lambda}^{\dot{A}}_{(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{A}}_{(i)}} - \frac{1}{2}\lambda^{A}_{(i)}\frac{\partial}{\partial\lambda^{A}_{(i)}}$$

  where h_i is the helicity of the state, $h_i = \pm 1$ in the case of gluons.
- Thus the n-particle amplitudes are also characterized by n helicities. For gluons these are ± 1 and the amplitude carries n sign indices,

$$\mathcal{A}(1,...,n) = \mathcal{A}^{-\cdots-\cdots+\cdots+}(1,...,n).$$

- It can be shown that $\mathcal{A}^{+\cdots+}(1,...,n) = 0$, $\mathcal{A}^{-+\cdots+}(1,...,n) = 0$,
- so that the simplest *maximal helicity violation (MHV)* amplitude is $\mathcal{A}^{MHV}(1,...,n) :=$

$$\mathcal{A}^{+\cdots+-_i+\cdots+-_j+\cdots+}(1,...,n) = \frac{<ij>^4}{<12>\ldots<n1>}\,\delta^4\left(\sum_i \lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)}\right)$$

  [Parke & Taylor, PRL86] ($<ij>\equiv<\lambda_{(i)}\lambda_{(j)}>=\epsilon_{AB}\lambda^{A}_{(i)}\lambda^{B}_{(j)}$).

BCFW deformations

- The BCFW recursion relations

$$\mathcal{A}_n = \sum_{\mathcal{I},h} \hat{\mathcal{A}}^h_{\mathcal{I}} \frac{1}{{P_{\mathcal{I}}}^2} \hat{\mathcal{A}}^{-h}_{\mathcal{J}}, \qquad \textit{where} \qquad \mathcal{I} \bigcup \mathcal{J} = (1, ..., n)$$

use the *on-shell* amplitudes depending on two deformed spinors, say

$$\lambda^A_{(n)} \mapsto \widehat{\lambda^A_{(n)}} = \lambda^A_{(n)} + z\lambda^A_{(1)}, \qquad \bar{\lambda}^{\dot{A}}_{(n)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(n)}} = \bar{\lambda}^{\dot{A}}_{(n)},$$

$$\lambda^A_{(1)} \mapsto \widehat{\lambda^A_{(1)}} = \lambda^A_{(1)} \,, \qquad \bar{\lambda}^{\dot{A}}_{(1)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(1)}} = \bar{\lambda}^{\dot{A}}_{(1)} - z\bar{\lambda}^{\dot{A}}_{(n)},$$

- which implies the deformation of 1st and n-th momenta

$$p^a_{(n)} \mapsto \widehat{p^a_{(n)}}(z) = p^a_{(n)} + zq^a \,, \qquad p^a_{(1)} \mapsto \widehat{p^a_{(1)}}(z) = p^a_{(1)} - z\bar{q}^a \,,$$

$$q^{A\dot{A}} = q^a \tilde{\sigma}^{A\dot{A}}_a = \lambda^A_{(1)} \bar{\lambda}^{\dot{A}}_{(n)} \quad \Rightarrow \quad q^a q_a = 0 \,, \qquad p^a_{(n)} q_a = 0 \,, \qquad p^a_{(1)} q_a = 0 \,.$$

The deformed momenta are generically complex but remain light-like,

$$\widehat{p^a_{(n)}}\, \widehat{p_{(n)a}} = 0 \,, \qquad \widehat{p^a_{(1)}}\, \widehat{p_{(1)a}} = 0 \,.$$

BCFW recurrent relations. Explicit form.

- The BCFW recurrent relations for tree amplitudes of D=4 gluons read

$$\mathcal{A}^{(n)}(p_1, p_2, \ldots; p_n) = \sum_h \sum_{l}^{n} \mathcal{A}_h^{(l+1)}(\widehat{p}_1(z_l); p_2; \ldots; p_l; \widehat{P_{\Sigma_l}}(z_l)) \times$$
$$\times \frac{1}{(P_{\Sigma_l})^2} \mathcal{A}_{-h}^{(n-l+1)}(-\widehat{P_{\Sigma_l}}(z_l), p_{l+1}; \ldots; \widehat{p}_n(z_l)) ,$$

where h is the helicity of intermediate state with $\widehat{P_{\Sigma_l}}(z_l)$,

$$P^a_{\Sigma_l} = -\sum_{m=1}^{l} p^a_m \qquad and \qquad \widehat{P^a_{\Sigma_l}}(z) = -\sum_{m=1}^{l} \widehat{p^a_m}(z)$$

- $\sum\limits_{l}$ is the sum over l and over distributions of particles among $\mathcal{A}_{\pm h}^{\{(l+1) \atop (n-l+1)}$.
- The specific l-dependent value of the complex parameter z,

$$z_l := P^a_{\Sigma_l} P_{\Sigma_l a} / 2 P^b_{\Sigma_l} q_b$$

- is such that $\boxed{(\widehat{P^a_{\Sigma_l}}(z_l))^2 = 0}$ $\Rightarrow$ r.h.s. contains on-shell amplitudes.

Superamplitudes and on-shell superfields for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA

- One can also collect the n-particle amplitudes of the fields of SYM (SUGRA) in the superfield amplitude (superamplitude)

$$\mathcal{A}(1;...;n)=\mathcal{A}(\lambda_{(1)},\bar{\lambda}_{(1)},\eta_{(1)};...;\lambda_{(n)},\bar{\lambda}_{(n)},\eta_{(n)})\ ,$$

depending on $\mathcal{N}=4$ ($\mathcal{N}=8$) fermionic $\eta^q_{(i)}=(\bar{\eta}_{q(i)})^*$ in fundamental rep. of $SU(4)$ ($SU(8)$), $q=1,...,4$ ($q=1,...,8$).

- This is possible because the on-shell states of the maximal SYM (SUGRA) multiplet can be collected in an **on-shell superfield**

$$\Phi(\lambda,\bar{\lambda},\eta^q)=f^{(+s)}+\eta^q\chi_q+\tfrac{1}{2}\eta^q\eta^p s_{pq}+\ldots+\tfrac{1}{\mathcal{N}!}\eta^q_1\ldots\eta^{q_\mathcal{N}}\epsilon_{q_1\ldots q_\mathcal{N}}f^{(-s)}\ ,$$

chiral superfield on an *on-shell superspace* of super-helicity $s=\frac{\mathcal{N}}{4}$,

$$\boxed{\hat{h}\Phi(\lambda,\bar{\lambda},\eta^q)=s\Phi(\lambda,\bar{\lambda},\eta^q)}\ ,\qquad \hat{h}:=-\tfrac{1}{2}\lambda^A\frac{\partial}{\partial\lambda^A}+\tfrac{1}{2}\bar{\lambda}^{\dot{A}}\frac{\partial}{\partial\bar{\lambda}^{\dot{A}}}+\tfrac{1}{2}\eta^q\frac{\partial}{\partial\eta^q}\ .$$

- The $\mathcal{N}=4$ (8) superamplitudes obey n superhelicity constraints

$$\hat{h}_{(i)}\mathcal{A}(\{\lambda_{(j)},\bar{\lambda}_{(j)},\eta^q_{(j)}\})=s\mathcal{A}(\{\lambda_{(j)},\bar{\lambda}_{(j)},\eta^q_{(j)}\})\ ,\qquad s=\frac{\mathcal{N}}{4}\ .$$

BCFW relations for superamplitudes

- In the BCFW-like recurrent relations for tree superamplitudes of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity [Brandhuber, Heslop, Travaglini, PRD 2008, Arkani-Hamed, Cachazo, Kaplan, JHEP 2010].

$$\mathcal{A}^{(n)}(k_1,\eta_1;\ldots;k_n,\eta_n)=$$
$$=\sum_l\int d^{\mathcal{N}}\eta\mathcal{A}^{(l+1)}_{z_l}(\widehat{k}_1,\widehat{\eta}_1;k_2,\eta_2;\ldots;k_l,\eta_l;\widehat{P_{\Sigma_l}}(z_l),\eta)\frac{1}{(P_{\Sigma_l})^2}\times$$
$$\times\mathcal{A}^{(n-l+1)}_{z_l}(-\widehat{P_{\Sigma_l}}(z_l),\eta;k_{l+1},\eta_{(l+1)};\ldots;k_{n-1},\eta_{n-1};\widehat{k}_n,\widehat{\eta}_n)\ .$$

- the deformations of the bosonic spinors

$$\widehat{\lambda^A_{(n)}}=\lambda^A_{(n)}+z\lambda^A_{(1)},\qquad \widehat{\bar\lambda^{\dot A}_{(1)}}=\bar\lambda^{\dot A}_{(1)}-z\bar\lambda^{\dot A}_{(n)},$$

- is supplenented by the deformation of fermionic $\eta^q=(\bar\eta_q)^*$,

$$\widehat{\eta^q_{(n)}}(z)=\eta^q_{(n)}+z\eta^q_{(1)}\ ,\qquad \widehat{\eta^q_{(1)}}(z)=\eta^q_{(1)}\ .$$

- Other new issues (w/r to bosonic BCFW): i) $\sum_l\mapsto\sum_l\int d^{\mathcal{N}}\eta$ and ii) $\widehat{\eta^q_{(n)}}(z)=\eta^q_{(n)}+z\eta^q_{(1)}$ which 'mixes' gluon and gluino amplitudes.

Outline

Spinor moving frame in D=11

- In D=4: $p_{\mu(i)}\sigma^{\mu}_{A\dot{A}} = 2\lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)} \quad \Leftrightarrow \quad p_{\mu(i)} = \lambda_{(i)}\sigma_{\mu}\bar{\lambda}_{(i)}$.
- Similarly, the light-like k_a of a massless 11D particle can be expressed by

$$\boxed{k_a\Gamma^a_{\alpha\beta} = 2\rho^{\#} v^{-}_{\alpha q} v^{-}_{\beta q}}\,, \qquad \boxed{\rho^{\#} v^{-}_{q}\tilde{\Gamma}_a v^{-}_{p} = k_a\delta_{qp}}\,,$$

  in terms of 'energy variable' $\rho^{\#}$ and
- a set of 16 **constrained** bosonic 32-component spinors $\boxed{v^{-}_{\alpha q}}$, $q,p = 1,...,16$, $\alpha = 1,...,32$ which can be identified with
  - D=11 spinor moving frame variables [Bandos, Zheeltukhin 92, Bandos 2006-2007]
  - 11D Lorentz harmonics [Galperin, Howe, Townsend NPB 93].
- Essentially, the constraints on $v^{-}_{\alpha q}$ are given by the above equations supplemented by $\boxed{v^{-}_{\alpha q}C^{\alpha\beta}v^{-}_{\beta p} = 0}$,
- and by the requirement that the rank of 32×16 matrix $v^{-}_{\alpha q}$ is = 16.

Spinor moving frame variables in D=11

- One can show that (roughly speaking) in the theory with local $SO(1,1)\otimes SO(9)$ symmetry, $\boxed{v_{\alpha q}^{-}}$ obeying the above constraints

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2\rho^{\#} v_{\alpha q}^{-} v_{\beta q}^{-} \; , \qquad v_q^{-}\tilde{\Gamma}_a v_p^{-} = u_a^{=}\delta_{qp} \; , \qquad v_{\alpha q}^{-} C^{\alpha\beta} v_{\beta q}^{-} = 0$$

$(u_a^{=} \equiv k_a/\rho^{\#})$ can be considered as homogeneous coordinates on $\mathbb{S}^9$, the celestial sphere of an 11D observer,

$$\boxed{\{v_{\alpha q}^{-}\} = \mathbb{S}^9} \qquad \left(\mathbb{S}^9 = \frac{SO(1,10)}{[SO(1,1)\otimes SO(9)] \subset\!\!\!\!\times K_9}\right)$$

Spinor moving frame and spinor helicity formalism

- One can check that, due to the above constraints the momentum k_a $(= \rho^{\#} u_a^{=})$ is light-like $\qquad \boxed{k_a k^a = 0}$
- and that $v_{\alpha q}^{-}$ and $v_q^{-\alpha} = -iC^{\alpha\beta} v_{\beta q}^{-}$ obey the Dirac equations

$$k_a \tilde{\Gamma}^{a\,\alpha\beta} v_{\beta q}^{-} = 0 \qquad \Leftrightarrow \qquad k_a \Gamma^a_{\alpha\beta} v_q^{-\beta} = 0 \; .$$

11D Spinor helicity formalism

- The 11D counterpart of the 10D spinor helicity variables of Caron-Huot and O'Connell are $\lambda_{\alpha q} = \sqrt{\rho^\#} v_{\alpha q}^-$;
- the 11D counterpart of the polarization spinor of the fermionic field is $\lambda_q^\alpha = \sqrt{\rho^\#} v_q^{-\alpha} = -iC^{\alpha\beta}\lambda_{\beta q}$ $(= (\lambda_q^\alpha)^*)$.
- The constraints on $v_{\alpha q}^-$ can be written in terms of λ_α

$$k_a \Gamma^a_{\alpha\beta} = 2\lambda_{\alpha q}\lambda_{\beta q}, \qquad \lambda_q \tilde{\Gamma}_a \lambda_p = k_a \delta_{qp} \qquad \lambda C \lambda = 0$$

- Then why we need $\rho^\#$ and $v_{\alpha q}^- = \lambda_{\alpha q}/\sqrt{\rho^\#}$?
  - The geometric and group theoretic meaning of $v_{\alpha q}^-$ is much more clear.
  - $\rho^\#$ and its canonically conjugate coordinate $x^=$ will play an important role in the construction of on-shell superfields and superamplitudes in D=10 and 11.
- In particular the D=11 counterpart of the on-shell superspace is

$$\Sigma^{(10|16)}: \qquad \{(x^=, v_{\alpha q}^-; \theta_q^-)\},$$

  with bosonic sector $\mathbb{R}\otimes\mathbb{S}^9$ including $\mathbb{R} = \{x^=\}$ and $\mathbb{S}^9 = \{v_{\alpha q}^-\}$.
- But where such seemingly strange spinor frame variables come from?

Vector frame attached to a light-like momentum

- Let us introduce a **moving frame matrix** or the matrix of **vector Lorentz harmonics** (or light-cone harmonics) [Sokatchev 86]

$$u_a^{(b)} = \left(\frac{1}{2}\left(u_a^{=} + u_a^{\#}\right), \, u_a^{I}, \frac{1}{2}\left(u_a^{\#} - u_a^{=}\right) \right) \in SO^{\uparrow}(1, D-1) .$$

- This obeys $u_a^{(b)} u^{a(c)} = \eta^{(a)(c)}$ (see [E. Sokatchev, 86,87]), i.e.

$$\begin{aligned} & u_a^{=} u^{a=} = 0 \, , \\ & u_a^{\#} u^{a\#} = 0 \, , \quad u_a^{=} u^{a\#} = 2 \, , \\ & u_a^{I} u^{a=} = 0 = u_a^{I} u^{a\#} \, , \quad u_a^{I} u^{aJ} = -\delta^{IJ} \end{aligned}$$

$$and \qquad \delta_a{}^b = \frac{1}{2} u_a^{=} u^{b\#} + \frac{1}{2} u_a^{\#} u^{b=} - u_a^{I} u^{bI} \, .$$

- Such a frame can be attached to a light-like momentum by setting

$$\boxed{k_a = \rho^{\#} u_a^{=}} \, .$$

Moving frame variables= $SO(1,D-1)/[SO(1,1)\otimes SO(D-2)] \subset\!\!\!\!\times K_{D-2} = \mathbb{S}^{D-2}$

- The splitting of $u_a^{(b)}$ is invariant under $[SO(1,1)\times SO(D-2)]$ and the relation $k_a = \rho^{\#} u_a^{=}$ is invariant under $H_B = [SO(1,1)\times SO(D-2)] \subset\!\!\!\!\times K_{D-2}$ where K_{D-2} is

$$u_a^{=} \mapsto u_a^{=}\,,$$
$$u_a^{I} \mapsto u_{a(i)}^{I} + \tfrac{1}{2} u_{a(i)}^{=} K^{\# I}\,,$$
$$u_a^{\#} \mapsto u_a^{\#} + \tfrac{1}{4} u_a^{=} (K^{\# I})^2 + u_a^{I} K^{\# I},$$

- using these symmetries as identificaation relations, we conclude that the set of harmonic variables parametrize a compact coset

$$\boxed{\{(u_a^{=}, u_a^{\#}, u_a^{I})\} = \frac{SO(1,D-1)}{[SO(1,1)\times SO(D-2)]\subset\!\!\!\!\times K_{D-2}} = \mathbb{S}^{D-2}}$$

[Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91].

- This can be also written as

$$\boxed{\{u_a^{=}\} = \mathbb{S}^{D-2}}.$$

Spinor moving frame = $\sqrt{\textit{moving frame}}$

- Spinor moving frame = $\sqrt{\textit{moving frame}}$ is defined by conditions of Lorentz invariance of D-dimensional Γ^a and also $C_{\alpha\beta}$ if such exists,
- i.e. is defined by a matrix $V \in Spin(1, D-1)$ which obeys

$$V\Gamma_b V^T = u_b^{(a)}\Gamma_{(a)} \,, \qquad V^T\tilde{\Gamma}^{(a)}V = \tilde{\Gamma}^b u_b^{(a)} \;,$$
$$VCV^T = C \,, \qquad \textit{for D in which} \quad \exists C \,.$$

- The $SO(1,1) \times SO(D-2)$ invariant splitting of the spinor moving frame matrix, corresponding to $u_b^{(a)} = (u_b^{=}, u_b^{\#}, u_b^{I})$, is

$$V_\alpha^{(\beta)} = \begin{pmatrix} v^{\;+}_{\alpha\dot{q}}, & v^{\;-}_{\alpha q} \end{pmatrix} \in Spin(1, D-1) \,,$$

where q and $\dot{q}$ are indices of the spinor representations of $SO(D-2)$, which can be different, like s-spinor and c-spinor in D=10,

$$D = 10: \qquad \alpha = 1, ..., 16 \,, \quad \dot{q} = 1, ..., 8 \,, \quad q = 1, ..., 8 \,,$$

or the same, as in D=11,

$$D = 11: \qquad \alpha = 1, ..., 32 \,, \quad q = \dot{q} = 1, ..., 16 \,, \quad v^{\;+}_{\alpha\dot{q}} \equiv v^{\;+}_{\alpha q} \,.$$

Spinor moving frame $= \sqrt{\textit{moving frame}}$

- The rectangular blocks of the spinor moving frame matrix, $v_{\alpha q}^{-}$ and $v_{\alpha \dot{q}}^{+}$ are called **spinor moving frame variables** or **spinor harmonics** (spinorial Lorentz harmonics).
- With the suitable representation for Γ–matrices, the constraints $V\Gamma_b V^T = u_b^{(a)}\Gamma_{(a)}$ and $V^T\tilde{\Gamma}^{(a)}V = \tilde{\Gamma}^b u_b^{(a)}$ can be split into

$$\boxed{u_a^{=}\Gamma^a_{\alpha\beta} = 2v_{\alpha q}{}^{-}v_{\beta q}{}^{-}}\;, \qquad \boxed{v_q^{-}\tilde{\Gamma}_a v_p^{-} = u_a^{=}\delta_{qp}},$$

$$, \qquad u_a^{\#}\Gamma^a_{\alpha\beta} = 2v_{\alpha\dot{q}}{}^{+}v_{\beta\dot{q}}{}^{+}\;, \qquad v_{\dot{q}}^{+}\tilde{\Gamma}_a v_{\dot{p}}^{+} = u_a^{\#}\delta_{\dot{q}\dot{p}}\;,$$

$$u_a^{I}\Gamma^a_{\alpha\beta} = 2v_{(\alpha|q}{}^{-}\gamma^I_{q\dot{q}}v_{|\beta)\dot{q}}{}^{+}\;, \qquad v_q^{-}\tilde{\Gamma}_a v_{\dot{p}}^{+} = u_a^{I}\gamma^I_{q\dot{p}}\;.$$

- These allow to state that $v_{\alpha q}^{-}$ is a square root of $u_a^{=}$ in the same sense as in D=4 one states λ_A "=" $\sqrt{p_a}$ $(p_\mu\sigma^\mu_{A\dot{A}} = 2\lambda_A\bar{\lambda}_{\dot{A}})$.
- *[In the above Eqs.: for D=11* $q,p \equiv \dot{q},\dot{p} = 1,...,16$ *are spinor indices of SO(9) and* $\gamma^I_{qp} = \gamma^I_{pq}$ *is the SO(9) gamma matrix,* $I = 1,...,9$*, while*
- *for D=10* $\gamma^I_{p\dot{q}} =: \tilde{\gamma}^I_{\dot{q}p}$ *are Klebsh-Gordan coefficients of SO(8),* $q,p = 1,...,8$ *are s-spinor (8s) indices,* $\dot{q},\dot{p} = 1,...,8$ *are c-spinor (8c) indices and I=1,.., 8].*

D=10 vs D=11 spinor helicity formalism

- The D=10 spinor helicity variables of Caron-Huot and O'Connell is

$$\lambda_{\alpha q} = \sqrt{\rho^{\#}} v_{\alpha q}^{-}$$

carrying 8s index, while the polarization spinor is

$$\lambda_{\dot{q}}^{\alpha} = \sqrt{\rho^{\#}} v_{\dot{q}}^{-\alpha}$$

which carries 8c spinor index of SO(8).

- It is constructed from the elements of the inverse spinor frame matrix

$$V_{(\beta)}{}^{\alpha} = \begin{pmatrix} v_q^{+\alpha} \\ v_{\dot{q}}^{-\alpha} \end{pmatrix} \in Spin(1, D-1) \; .$$

- In contrast to 11D, where the polarization vector actually coincides with the spinor helicity variable

$$\lambda_q^{\alpha} = \sqrt{\rho^{\#}} v_q^{-\alpha} = -iC^{\alpha\beta}\lambda_{\beta q} \; .$$

On shell fields of D=10 SYM in spinor frame form of spinor helicity formalism

- Thus the general solution of the massless Dirac (Weyl) equation

$$D = 10: \quad \chi^{\alpha} = v_{\dot{q}}^{-\alpha}\psi_{\dot{q}}\,, \qquad \dot{q} = 1, ..., 8\,,$$

  is characterized by a fermionic SO(8) c-spinor $\psi_{\dot{q}}$.

- The polarization vector of the vector field can be identified with u_a^I so that the on-shell field strength of the (D=10) gauge field

$$D = 10: \quad F_{ab} = k_{[a}u_{b]}{}^{I}\, w^{I}\,, \qquad a = 0, 1, ..., 9\,, \qquad I = 1, ..., 8$$

  is characterized by an SO(8) vector w^I.

- The on-shell d.o.f.'s of SYM $\leftrightarrow$ $w^I = w^I(\rho^{\#}, v_{\alpha q}^{-})$, $\psi_{\dot{q}} = \psi_{\dot{q}}(\rho^{\#}, v_{\alpha q}^{-})$ or, making Fourier transform w/r to $\rho^{\#}$, $w^I(x^{=}, v_q^{-})$ and $\psi_q(x^{=}, v_q^{-})$.

- Supersymmetry acts on these 9d fields by

$$\delta_{\epsilon}\psi_{\dot{q}} = \epsilon^{-q}\gamma^{I}_{q\dot{q}}\, w^{I}\,, \qquad \delta_{\epsilon} w^{I} = 2i\epsilon^{-q}\gamma^{I}_{q\dot{q}}\partial_{=}\psi_{\dot{q}},$$

$$\textit{where} \qquad \epsilon^{-q} = \epsilon^{\alpha} v_{\alpha q}^{-}\,.$$

On shell fields of D=11 SUGRA in spinor frame/spinor helicity formalism

- The linearized on-shell field strength of 3-form gauge field

$$D=11: \quad F_{abcd}=k_{[a}u_b{}^I u_c{}^J u_{d]}{}^K\,\Phi_{IJK}\,, \qquad a=0,1,...,10, \quad I=1,...,9\,,$$

  is expressed in terms of antisymmetric SO(9) tensor Φ_{IJK} $(=A_{IJK})$.

- Its superpartners, γ–traceless Ψ_{Iq} and symmetric and traceless h_{IJ}, which can be used to write the general solutions of the linearized equations for 11D graviton and gravitino fields,

$$D=11: \qquad \psi^{\alpha}_{ab}=k_{[a}u^I_{b]}v_q^{-\alpha}\Psi_{Iq}\,, \qquad \gamma^I_{qp}\Psi_{Ip}=0\,,$$
$$h_{ab}=u^I_{(a}u^J_{b)}h_{IJ}\,, \qquad h_{II}=0$$

  $(R_{ab}{}^{cd}\;=k_{[a}u^I_{b]}k^{[c}u^{d]J}h_{IJ})$.

- These fields will appear as independent components of a constrained on-shell superfield.

Outline

On-shell superspace

- The constrained on-shell superfields of 10D SYM and 11D SUGRA
- are functions on the on-shell superspaces (with $\mathcal{N}=4$ and $\mathcal{N}=8$)

$$\Sigma^{((D-1)|2\mathcal{N})} = \{x^{=}, v_{\alpha q}^{-}, \theta_q^{-}\} , \quad \alpha = 1, ..., 4\mathcal{N} , \quad q = 1, ..., 2\mathcal{N} ,$$

- or on their 'fully momentum' versions $\tilde{\Sigma}^{((D-1)|2\mathcal{N})} = \{\rho^{\#}, v_{\alpha q}^{-}, \theta_q^{-}\}$ with bosonic bodies $\mathbb{R}_{+}^{1} \times \mathbb{S}^{(D-2)}$.
- SUSY acts on the coordinates of $\Sigma^{((D-1)|2\mathcal{N})}$

$$\delta_\epsilon x^{=} = 2i\theta_q^{-} \; \epsilon^\alpha v_{\alpha q}^{-} , \qquad \delta_\epsilon \theta_q^{-} = \epsilon^\alpha v_{\alpha q}^{-} , \qquad \delta_\epsilon v_{\alpha q}^{-} = 0 .$$

- $\Rightarrow \Sigma^{((D-1)|2\mathcal{N})}$ can be considered as an invariant subsuperspace of Lorentz harmonic superspace $\Sigma^{(2(D-2)|4\mathcal{N})} = \{X^\mu, \Theta^\alpha; v_{\alpha q}^{-}, v_{\alpha \dot{q}}^{+}\}$:

$$x^{=} = X^a u_a^{=} , \qquad \theta_q^{-} = \Theta^\alpha v_{\alpha q}^{-} .$$

- On-shell superfields can be treated as special Lorentz harmonic superfields depending on $x^{=} = X^a u_a^{=}$, $\theta_q^{-} = \Theta^\alpha v_{\alpha q}^{-}$ and $v_{\alpha q}^{-}$ only,
- which obey some equations making them (one-to-one related with the) solutions of the superfield equations of 10D SYM and 11D SUGRA.

On-shell superfield description of D=10 SYM

- The main on-shell superfield of D=10 SYM is [A. Galperin, P. Howe, P. Townsend NPB1993] a fermionic c-spinor superfield $\Psi_{\dot{q}}$ obeying

$$D^+_q \Psi_{\dot{q}} = \gamma^I_{q\dot{q}} V^I \, , \qquad D^+_q = \frac{\partial}{\partial \theta^-_q} + 2i\theta^-_q \frac{\partial}{\partial x^=} \, .$$

- The consistency of this eq. requires

$$D^+_q V^I = 2i\gamma^I_{q\dot{q}} \partial_= \Psi_{\dot{q}} \, , \qquad q = 1, ..., 8 \, , \quad \dot{q} = 1, .., 8 \, , \quad I = 1, .., 8$$

- $\Rightarrow$ there are no other independent components in the constrained on-shell superfield $\Psi_{\dot{q}}(x^=, \theta^-_q, v_{\alpha q}{}^-)$, but $\psi_{\dot{q}} = \psi_{\dot{q}}|_0$ and $w^I = V^I|_0$.

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Indeed,

$$\Psi_{\dot{q}}(x^=, v^-_q; \theta^-_q) = \psi_{\dot{q}}(x^=, v^-_q) + \theta^-_q \gamma^I_{q\dot{q}} w^I(x^=) +$$

$$+ \sum_{k=1}^{4} \frac{(-i)^k \, (2k-1)!!}{(2k)!! \, (2k)!} (\theta^- \gamma^{I_{k-1} I_k} \theta^-) \ldots (\theta^- \gamma^{I_1 I_2} \theta^-) \, (\gamma^{I_1 I_2} \ldots \gamma^{I_{k-1} I_k})_{\dot{q}\dot{p}} (\partial_=)^k \psi_{\dot{p}} +$$

$$+ \sum_{k=1}^{3} \frac{(-i)^k \, (2k)!!}{(2k+1)!! \, (2k+1)!} (\theta^- \tilde{\gamma}^{I_1 I_2} \theta^-) \ldots (\theta^- \tilde{\gamma}^{I_{k-1} I_k} \theta^-) (\tilde{\gamma}^{I_1 I_2} \ldots \tilde{\gamma}^{I_{k-1} I_k} \tilde{\gamma}^I \theta^-)_{\dot{q}} (\partial_=)^k w^I .$$

On-shell superfields of 11D SUGRA

- In [A. Galperin, P. Howe, P. Townsend NPB1993] the linearized 11D supergravity was described by a bosonic superfield $\Phi^{IJK} = \Phi^{[IJK]}(x^{=}, \theta_q^-, v_{\alpha q}{}^-)$ which obeys

$$D_q^+ \Phi^{IJK} = 3i\gamma^{[IJ}_{qp}\Psi^{K]}_p\,, \qquad \gamma^I_{qp}\Psi^I_p = 0\,, \qquad \begin{cases} I,J,K = 1,...,9 \\ q,p = 1,...,16 \end{cases}$$

where $\gamma^I_{qp} = \gamma^I_{pq}$ are d=9 Dirac matrices, $\gamma^I\gamma^J + \gamma^J\gamma^I = \delta^{IJ}\mathbb{I}_{16\times 16}$, and

$$D_q^+ = \partial_q^+ + 2i\theta_q^-\partial_{=} \equiv \frac{\partial}{\partial\theta_q^-} + 2i\theta_q^-\frac{\partial}{\partial x^{=}}$$

obeying the d=1, $N=16$ supersymmetry algebra

$$\{D_q^+, D_p^+\} = 4i\delta_{qp}\partial_{=}\ .$$

On-shell superfield equations of linearized D=11 SUGRA

- The consistency of $D_q^+\Phi^{IJK} = 3i\gamma_{qp}^{[IJ}\Psi_p^{K]}$ requires, besides $\gamma_{qp}^I\Psi_p^I = 0$, that

$$D_q^+\Psi_p^I = \frac{1}{18}\left(\gamma_{qp}^{IJKL} + 6\delta^{I[J}\gamma_{qp}^{KL]}\right)\partial_{=}\Phi^{JKL} + 2\partial_{=}H_{IJ}\gamma_{qp}^J \; ,$$

  with symmetric traceless $SO(9)$ tensor superfield $H_{IJ} = H_{((IJ))}$, obeying

$$D_q^+H_{IJ} = i\gamma_{qp}^{(I}\Psi_p^{J)}, \qquad H_{IJ} = H_{JI}, \quad H_{II} = 0 \; .$$

- These superfield equations (actually any of these three) can be considered as a (part of a) counterpart of superhelicity constraint $\hat{h}\Phi = h\Phi$ imposed on the D=4 on-shell superfield.

On-shell superfield equations of linearized D=11 SUGRA

- It is convenient to collect all the on-shell superfields in one object

$$\Psi_Q(x^{=}, v^{-}_{\alpha q}; \theta^{-}_q) = \{ \Psi_{Iq} , \Phi_{[IJK]} , H_{((IJ))} \} ,$$

with multi-index Q taking 128(=144-16) 'fermionic' and 128=84+44 'bosonic values',

$$Q = \{ Iq , [IJK] , ((IJ)) \}$$

(gamma-tracelessness and tracelessness are implied!),

- and to write all the equations for them,

$$D^{+}_q \Psi^{I}_p = \frac{1}{3} \left(\gamma^{IJKL}_{qp} + 6\delta^{I[J} \gamma^{KL]}_{qp} \right) \partial_{=} \Phi^{JKL} + 2\partial_{=} H_{IJ} \gamma^{J}_{qp} ,$$

$$D^{+}_q \Phi^{IJK} = 3i\gamma^{[IJ}_{qp} \Psi^{K]}_p , \qquad D^{+}_q H_{IJ} = i\gamma^{(I}_{qp} \Psi^{J)}_p ,$$

in the unique form

$$D^{+}_q \Psi_Q = \Delta_{Q\, qP} \Psi_P .$$

Fourier transform of the linearized 11D SUGRA equations

- After making Fourier transform

$$\Psi_Q(\rho^{\#}, v_{\alpha q}^{-}; \theta_q^{-}) = \frac{1}{2\pi} \int dx^{=} \exp(i\rho^{\#} x^{=})\, \Psi_Q(x^{=}, v_{\alpha q}^{-}; \theta_q^{-})$$

- the superfields obey the same $D_q^+ \Psi_Q = \Delta_{Q\,qP} \Psi_P$ but with $\partial_{=} \mapsto -i\rho^{\#}$,

$$D_q^+ = \partial_q^+ + 2\rho^{\#}\theta_q^- \ .$$

- All $\Delta_{Q\,qP}$ are now algebraic, in particular

$$\boxed{D_q^+ \Psi_p^I = -\frac{i\rho^{\#}}{3} \left(\gamma^{IJKL} + 6\delta^{I[J}\gamma^{KL]}\right)_{qp} \Phi^{JKL} - 2i\rho^{\#} H_{IJ}\gamma^J_{qp}} \ .$$

- Our 11D superamplitudes should obey a certain generalization of these equations, $D_q^+ \Psi_Q = \Delta_{Q\,qP} \Psi_P$.
- The most convenient way is to start from one of the bosonic superamplitudes.

Outline

10D superamplitudes

- The on-shell n-particle superamplitudes are functions on a direct product of n copies of the on-shell superspace.
- The basic superamplitude of 10D SYM

$$\mathcal{A}^{(n)}_{I_1\ldots I_n}(k_1,\theta_1^-;\ldots;k_n,\theta_n^-) \equiv \mathcal{A}^{(n)}_{I_1\ldots I_n}(\rho_1^{\#};v_{q1}^-;\theta_{q1}^-;\ldots;\rho_n^{\#};v_{qn}^-;\theta_{qn}^-) \ ,$$

carry n 'bosonic' **8v** indices of SO(8) and obeys

$$\boxed{D^+_{qj}\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n} = 2\rho_j^{\#}\,\gamma^{I_j}{}_{q\dot q}\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot q I_{j+1}\ldots I_n}}, \qquad D^+_{qj} = \frac{\partial}{\partial\theta^-_{qj}} + 2\rho_j^{\#}\theta^-_{qj} \ .$$

- Selfconsistency of this equation requires equations for $\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot q I_{j+1}\ldots I_n}$ and for amplitudes with higher number of fermions.
- It is convenient to introduce a notation with multi-indices $Q_j = \{\dot q_j, I_j\}$ and resume all these equations in one

$$D^+_{qj}\mathcal{A}_{Q_1\ldots Q_j\ldots Q_j} = (-)^{\Sigma_j}\Delta_{Q_j\,qP_j}\mathcal{A}_{Q_1\ldots P_j\ldots Q_j}.$$

- $\Delta_{Q_j\,qP_j}$ can be read off the equations for on-shell superfields, $\Delta_{Iq\dot q} = 2\rho_j^{\#}\,\gamma^{I_j}{}_{q\dot q}$ etc.

11D superamplitudes

- The on-shell n-particle scattering amplitudes of 11D SUGRA

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta_1^-;\ldots;k_n,\theta_n^-)\equiv\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(\rho_1^\#;v_{q1}^-;\theta_{q1}^-;\ldots;\rho_n^\#;v_{qn}^-;\theta_{qn}^-)\;, \qquad ,$$

carry n multi-indices $Q_l=\{I_lq_l\,,[I_lJ_lK_l]\,,((I_lJ_l))\}$ and obey

$$\gamma^{I_l}_{p_lq_l}\mathcal{A}_{\ldots I_{(l)}q_{(l)}\ldots}=0,$$

$$D^+_{q(l)}\mathcal{A}_{\ldots Q_{(l)}\ldots}=(-)^{\Sigma_l}\Delta_{Q_l\,qP_{(l)}}\mathcal{A}_{\ldots P_{(l)}\ldots},$$

- $\Delta_{Q_j\,qP_j}$ can be read off eqs. for on-shell superfields,
- and $\Sigma_l=\#$ of fermionic, I_jq_j, indices among $Q_1,\ldots Q_{(l-1)}$, *i.e.*

$$\Sigma_l=\sum_{j=1}^{l-1}\frac{(1-(-)^{\varepsilon(Q_j)})}{2}\;, \qquad \begin{cases}\varepsilon([I_jJ_jK_j])=0=\varepsilon(\,((I_jJ_j))\,)\;,\\ \varepsilon(I_jq_j)=1\;.\end{cases}$$

- In particular, when $Q_l=I_lp_l$, this equation reads

$$(-)^{\Sigma_l}D^{+(l)}_{q_l}\mathcal{A}^{(n)}_{Q_1\ldots\,I_lp_l\,\ldots Q_n}=-i\rho^\#_{(l)}\gamma_{J_l\,qp}\mathcal{A}^{(n)}_{Q_1\ldots((I_lJ_l))\ldots Q_n}-$$
$$-\tfrac{i}{18}\rho^\#_{(l)}\left(\gamma^{I_lJ_lK_lL_l}_{qp}+6\delta^{I_l[J_l}\gamma^{K_lL_l]}_{qp}\right)\mathcal{A}^{(n)}_{Q_1\ldots[J_lK_lL_l]\ldots Q_n}\;.$$

BCFW-type relations for constrained 11D and 10D superamplitudes

- Candidate BCFW-type relations for above described constrained superamplitudes in D=11 and D=10 have been obtained in [PRL 2017] and analysed in [JHEP 2018].
- As they relate on-shell superamplitudes, we need to define a deformation of our spinor helicity (spinor frame) and fermionic variables which result in a shift of, say

$$\widehat{k^a_{(1)}} = k^a_{(1)} - zq^a \,, \qquad \widehat{k^a_{(n)}} = k^a_{(n)} + zq^a \,, \qquad z \in \mathbb{C},$$
$$q_a q^a = 0 \,, \qquad q_a k^a_{(1)} = 0 \,, \qquad q_a k^a_{(n)} = 0 \,.$$

Generalized BCFW deformations in D=11

- In D=11 and D=10 that results from

$$\widehat{v^{\;-}_{\alpha q(n)}} = v^{\;-}_{\alpha q(n)} + z\, v^{\;-}_{\alpha p(1)}\, \mathbb{M}_{pq}\, \sqrt{\rho^{\#}_{(1)}/\rho^{\#}_{(n)}}\;,$$
$$\widehat{v^{\;-}_{\alpha q(1)}} = v^{\;-}_{\alpha q(1)} - z\, \mathbb{M}_{qp}\, v^{\;-}_{\alpha p(n)}\, \sqrt{\rho^{\#}_{(n)}/\rho^{\#}_{(1)}}$$

where $\mathbb{M}_{qp} = -2\, q^a\, (v^{\;-}_{q(1)}\, \tilde{\Gamma}_a v^{\;-}_{p(n)})\, \sqrt{\rho^{\#}_{(1)}\rho^{\#}_{(n)}}/(k_{(1)}k_{(n)})$ is nilpotent

$$\boxed{\mathbb{M}_{rp}\mathbb{M}_{rq} = 0}, \boxed{\mathbb{M}_{qr}\mathbb{M}_{pr} = 0}.$$

- The deformed $\widehat{v^{\;-}_{\alpha q(1)}}$ and $\widehat{v^{\;-}_{\alpha q(n)}}$ are complex but obey the characteristic constraints $\boxed{\widehat{u^{=}_{a(i)}}\Gamma^a_{\alpha\beta} = 2\widehat{v^{\;-}_{\alpha q(i)}}\,\widehat{v_{\beta q(i)}^{\;-}}}$ and $\boxed{\widehat{v^{\;-}_{q(i)}}\tilde{\Gamma}_a \widehat{v^{\;-}_{p(i)}} = \widehat{u^{=}_{a(i)}}\delta_{qp}}$!
- The deformation of the fermionic variables reads

$$\widehat{\theta^{-}_{p(n)}} = \theta^{-}_{p(n)} + z\, \theta^{-}_{q(1)}\, \mathbb{M}_{qp}\, \sqrt{\rho^{\#}_{(1)}/\rho^{\#}_{(n)}}\;,$$
$$\widehat{\theta^{-}_{q(1)}} = \theta^{-}_{q(1)} - z\, \mathbb{M}_{qp}\, \theta^{-}_{p(n)}\, \sqrt{\rho^{\#}_{(n)}/\rho^{\#}_{(1)}}\;.$$

11D BCFW

BCFW-type recurrent relations for tree 11D superamplitudes [PRL 2017] are

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};\ldots;k_n,\theta^-_{(n)}) =$$

$$= \sum_{l=2}^{n} \frac{(-)^{\Sigma(l+1)}}{64(\widehat{\rho}^{\#}(z_l))^2} D^+_{q(z_l)} \Big(\mathcal{A}^{(l+1)}_{z_l\; Q_1\ldots Q_l\; Jp}(\widehat{k}_1,\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\ldots;k_l,\theta^-_{(l)};\widehat{P}_l(z_l),\theta^-) \times$$

$$\times \frac{1}{(P_l)^2} \overleftrightarrow{D}^+_{q(z_l)} \mathcal{A}^{(n-l+1)}_{z_l\; Jp\; Q_{l+1}\ldots Q_n}(-\widehat{P}_l(z_l),\theta^-;k_{l+1},\theta^-_{(l+1)};\ldots;k_{n-1},\theta^-_{(n-1)};\widehat{k}_n,\widehat{\theta^-_{(n)}}) \Big)\Big|_{\theta^-=0} .$$

- where $P^a_l = -\sum_{m=1}^{l} k^a_m\,,\quad \widehat{P}^a_l(z) = -\sum_{m=1}^{l<n} \widehat{k}^a_m(z) = P^a_l - zq^a$ and $\boxed{z_l := \frac{P^a_l P_{l\,a}}{2P^b_l q_b}}$ with q^a obeying $q^2=0$, $q\cdot k_1 = 0$, $q\cdot k_n = 0$
- One can find that $q^a = -\sqrt{\rho^{\#}_1\rho^{\#}_n}\, v^-_{q(1)}\tilde{\Gamma}^a \mathbb{M}_{qp} v^-_{p(n)}/32$ with $\mathbb{M}\mathbb{M}^T = 0$.
- Actually, the bosonic arguments of the on-shell amplitudes are $\rho^{\#}_{(i)}$ and $v^-_{\alpha q(i)}$ from $k_{a(i)}\Gamma^a_{\alpha\beta} = 2\rho^{\#}_{(i)} v^-_{\alpha q(i)} v^-_{\beta q(i)}$ and $v^-_{q(i)}\tilde{\Gamma}^a v^-_{p(i)} = k_{a(i)}\delta_{qp}/\rho^{\#}_{(i)}$.

11D BCFW

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};\ldots;k_n,\theta^-_{(n)}) =$$

$$= \sum_{l=2}^{n} \frac{(-)^{\Sigma_{(l+1)}}}{64(\widehat{\rho}^{\#}(z_l))^2} D^+_{q(z_l)} \Big(\mathcal{A}^{(l+1)}_{z_l\; Q_1\ldots Q_l\; Jp}(\widehat{k}_1,\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\ldots;k_l,\theta^-_{(l)};\widehat{P}_l(z_l),\theta^-) \times$$

$$\times \frac{1}{(P_l)^2} \overleftrightarrow{D}^+_{q(z_l)} \mathcal{A}^{(n-l+1)}_{z_l\; Jp\; Q_{l+1}\ldots Q_n}(-\widehat{P}_l(z_l),\theta^-;k_{l+1},\theta^-_{(l+1)};\ldots;k_{n-1},\theta^-_{(n-1)};\widehat{k}_n,\widehat{\theta^-_{(n)}}) \Big)\Big|_{\theta^-=0} .$$

- Actually, the bosonic arguments of the on-shell amplitudes are $\rho^{\#}_{(i)}$ and $v^-_{\alpha q(i)}$ from $k_{a(i)}\Gamma^a_{\alpha\beta} = 2\rho^{\#}_{(i)} v^-_{\alpha q(i)} v^-_{\beta q(i)}$ and $v^-_{q(i)}\tilde{\Gamma}^a v^-_{p(i)} = k_{a(i)}\delta_{qp}/\rho^{\#}_{(i)}$.
- and $\pm\widehat{P}_l{}^a(z_l)$ should be also understood as $v^-_{\alpha q P_l}(z_l)$ and $\pm\rho^{\#}_{P_l}(z_l)$

$$\widehat{P_l{}^a}(z_l)\Gamma_{a\alpha\beta} = 2\rho^{\#}_{P_l}\, v^-_{\alpha q P_l} v^-_{\beta q P_l} \; , \qquad \widehat{P}_l{}^a(z_l)\delta_{qp} = \rho^{\#}_{P_l}\, v^-_{q\, P_l}\tilde{\Gamma}^a v^-_{p\, P_l} \; .$$

- Finally, $D^+_{q(z_l)}$ is the covariant derivative with respect to θ^-_q, $D^+_{q(z_l)} = \frac{\partial}{\partial\theta^-_q} + 2\rho^{\#}_{P_l}\theta^-_q$.

Exotic measure in 11D BCFW

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(\ldots) = \sum_{l=2}^{n}(\ldots)\times$$

$$\times D^+_q \left(\mathcal{A}^{(l+1)}_{z_l\; Q_1\ldots Q_l\; Jp}(\ldots;\widehat{P}_l,\theta^-)\frac{1}{(P_l)^2}\overleftrightarrow{D}^+_q \mathcal{A}^{(n-l+1)}_{z_l\; Jp\; Q_{l+1}\ldots Q_n}(-\widehat{P}_l,\theta^-;\ldots)\right)_{\theta^-=0} .$$

- The expression $D^+_q(\mathfrak{B}_q)|_{\theta^-=0}$ can be considered as Grassmann integration with an exotic fermionic measure.
- Such type of measure was used by Mario Tonin in 1990 to write a superfield (STV-type) action for heterotic superstring in D=10.
- [Tonin:1991ii] M. Tonin, “World sheet supersymmetric formulations of Green-Schwarz superstrings,” Phys.Lett. B **266** (1991) 312.
- [Tonin:1991ia] M. Tonin, “kappa symmetry as world sheet supersymmetry in D = 10 heterotic superstring,” Int. J. Mod. Phys. A **7** (1992) 6013.

Issues

- The further study of these candidate 10D and 11D BCFW relations [JHEP 2018] made manifest some issues
- one of which is the dependence of the calculated amplitudes on the deformation vector q^a.
- Such a dependence seems to characteristic also for other approaches to higher dimensional generalizations of BCFW in higher dimensions [Arkani-Hamed + Kaplan 2008, Cheung 2008], even for bosonic amplitudes (while for D=4 this is completely fixed by its characteristic properties $0 = q^a p_{a1} = q^a p_{an} = q^a q_a$) and a propositions to make some *ad hoch* choices were made.
- However, in my opinion, this q-dependence should be either improved or understood better.

Other approaches?

- Let me also stress that BCFW is not a unique approach for (super)amplitude calculations so the other methods to calculate our 10D SYM and 11D SUGRA constrained superamplitudes can be developed.

Outline

- Let us start from constrained on-shell superfields of 10D SYM

$$D^+_q W^I = 2i\gamma^I_{q\dot{q}}\Psi_{\dot{q}} \qquad \Rightarrow \qquad D^+_q \Psi_{\dot{q}} = \gamma^I_{q\dot{q}}\,\partial_= W^I \ .$$

Breaking $SO(8) \mapsto SO(6) \otimes SO(2) \approx SU(4) \otimes U(1)$, we can split the vector representation $\mathbf{8}_v$ of SO(8) on **6+1+1** of $SO(6)$,

$$W^I = (W^{\check{I}}, W^7, W^8), \qquad \check{I} = 1, ..., 6 \ ,$$

and introducing

$$\Phi = \frac{W^7 - iW^8}{2}, \qquad \bar{\Phi} = \frac{W^7 + iW^8}{2}, \qquad \Psi_q = \gamma^8_{q\dot{q}}\Psi_{\dot{q}} \ ,$$

we find that the above equation implies

$$D^+_q \Phi \;=\; \left(\delta_{qp} + i(\gamma^7\tilde{\gamma}^8)_{qp}\right)\Psi_p =: 2\mathcal{P}_{+qp}\Psi_p \ ,$$
$$D^+_q \bar{\Phi} \;=\; -\left(\delta_{qp} - i(\gamma^7\tilde{\gamma}^8)_{qp}\right)\Psi_p =: 2\mathcal{P}_{-qp}\Psi_p \ .$$

- It is important to notice that the matrices $\mathcal{P}^{\pm}_{qp} = \frac{1}{2}\left(\delta_{qp} \pm i(\gamma^7\tilde{\gamma}^8)_{qp}\right)$, are orthogonal projectors

$$\mathcal{P}^+\mathcal{P}^+ = \mathcal{P}^+ , \qquad \mathcal{P}^-\mathcal{P}^- = \mathcal{P}^- , \qquad \mathcal{P}^+\mathcal{P}^- = 0$$
$$\mathcal{P}^+ + \mathcal{P}^- = \mathbb{I} , \qquad (\mathcal{P}^+)^* = \mathcal{P}^- ,$$

and hance that $D^+_q\Phi = 2\mathcal{P}_{+qp}\Psi_p$ implies

$$\mathcal{P}_{-qp}D^+_p\Phi = 0 , \qquad \mathcal{P}_{+qp}D^+_p\bar{\Phi} = 0.$$

Furthermore, as the projectors $\mathcal{P}^+$ and $\mathcal{P}^-$ are complementary and complex conjugate, we can introduce complex 8×4 matrix $w_q{}^A$ and its complex conjugate $\bar{w}_{qA}$ such that

$$\mathcal{P}^+_{qp} = 2w_q{}^A\bar{w}_{pA} , \qquad \mathcal{P}^-_{qp} = 2\bar{w}_{qA}w_p{}^A.$$

In terms of these rectangular blocks the above equations can be written as chirality (analyticity) conditions

$$\bar{D}^+_A\Phi = 0 , \qquad D^{+A}\bar{\Phi} = 0 , \qquad \text{with} \qquad \bar{D}^+_A = \bar{w}_{pA}D^+_q , \qquad D^+_A = w_q{}^A D^+_q .$$

- The construction may be made SO(8) invariant by introducing a 'bridge' coordinates parametrizing $SO(8)/[SU(4)\otimes U(1)]$ coset: the $SO(8)$ valued matrix

$$U_I^{(J)} = \left(U_I^{\check{J}}, U_I^{(7)}, U_I^{(8)}\right) = \left(U_I^{\check{J}}, \tfrac{1}{2}\left(U_I + \bar{U}_I\right), \tfrac{1}{2i}\left(U_I - \bar{U}_I\right)\right) \in SO(8)$$

with its elements (internal vector harmonics) obeying

$$U_I U_I = 0\,, \qquad \bar{U}_I \bar{U}_I = 0\,, \qquad U_I \bar{U}_I = 2\,,$$
$$U_I U_I^{\check{J}} = 0\,, \qquad \bar{U}_I U_I^{\check{J}} = 0\,, \qquad U_I^{\check{J}} U_I^{\check{K}} = \delta^{\check{J}\check{K}}\,.$$

Then the SO(8) covariant projectors

$$\mathcal{P}^+_{qp} = \frac{1}{2}\left(\delta_{qp} + i(\gamma^I\tilde{\gamma}^J)_{qp} U_I^{(7)} U_J^{(8)}\right) = \frac{1}{4}\gamma^I\tilde{\gamma}^J \bar{U}_I U_J\,,$$
$$\mathcal{P}^-_{qp} = \frac{1}{2}\left(\delta_{qp} - i(\gamma^I\tilde{\gamma}^J)_{qp} U_I^{(7)} U_J^{(8)}\right) = \frac{1}{4}\gamma^I\tilde{\gamma}^J U_I \bar{U}_J$$

and w, $\bar{w}$ are spinir internal harmonics, the 'square roots' of U and $\bar{U}$.
- Similarly we can proceed with superamplitudes.

Little group SO(D-2)$\mapsto$ SO(D-4) tiny group

- Thus there exists a possibility to construct an alternative, analytic superfield formalism [JHEP 2018 =hep-th/1705.09550].
- The price to pay is that the little group symmetry $SO(D-2)_i$ is broken (spontaneously) to the 'tiny group' $SO(D-4)$ ($\subseteq SU(\mathcal{N})$).
- An analytic superamplitude has a superfield structure very similar to its 4D cousin, but depend on another set of bosonic variables. These are:
- D=10 or D=11 spinor helicity variables: densities $\rho^{\#}_i$ and $v^-_{\alpha qi}$

$$\{v^-_{\alpha qi}\} = \left(\frac{Spin(1,D-1)}{[SO(1,1)\otimes Spin(D-2)] \subset\!\!\!\!\times K_{D-2}}\right)_i ,$$

and internal frame or internal harmonic variables

$$\{w^A_{qi}, \bar{w}_{Aqi}\} = \left(\frac{Spin(D-2)}{Spin(D-4)\otimes U(1)}\right)_i$$

[*Harmonic variables, SU(2)/U(1), SU(3)/(U(1)XU(1)),... :*
[Galperin, Ivanov, Kalitsin, Ogievetsky, Sokatchev=GIKOS CQG 84,84],
[Galperin, Ivanov, Ogievetsky, Sokatchev, *Harmonic superspace*, 2001]].

$\frac{SO(D-2)}{SO(D-4)\otimes U(1)}$ harmonic variables

- This internal frame or internal harmonic variables

$$\{w^A_{qi}, \bar{w}_{Aqi}\} = \left(\frac{Spin(D-2)}{Spin(D-4)\otimes U(1)}\right)_i \; ,$$

obey, besides

$$\Psi_{q\dot{p}} := \gamma^I_{q\dot{p}} U_I = 2\bar{w}_{qA} w^A_{\dot{p}} \; , \qquad \bar{\Psi}_{q\dot{p}} := \gamma^I_{q\dot{p}} \bar{U}_I = 2 w^A_q \bar{w}_{\dot{p}A} \; .$$

and $\Psi^{\check{J}}_{q\dot{p}} := \gamma^I_{q\dot{p}} U^{\check{J}}_I = i w^A_q \sigma^{\check{J}}_{AB} w^B_{\dot{p}} + i\bar{w}_{qA}\tilde{\sigma}^{\check{J}AB}\bar{w}_{\dot{p}B}$, also

$$\bar{w}_{qB} w_q{}^A = \delta_B{}^A \; , \qquad w_q{}^A w_q{}^B = 0 \; , \qquad \bar{w}_{qA}\bar{w}_{qB} = 0 \; .$$

- This reflects that for $D=10$: $Spin(D-4) = Spin(6) = SU(4)$, and for $D=11$: $Spin(D-4) = Spin(7) \subset SU(8)$.
- In the above constraints U_I, $\bar{U}_I$ and $U_I{}^{\check{J}}$ form the vector internal frame

$$U^{(J)}_I = \left(U_I{}^{\check{J}}, \tfrac{1}{2}\left(U_I + \bar{U}_I\right), \tfrac{1}{2i}\left(U_I - \bar{U}_I\right)\right) \; \in \; SO(D-2) \; .$$

Analytic superamplitude of 10D SYM

- We start with the basic $\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n}$ obeying

$$D^{+(j)}_{q_j}\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n}=2\rho^{\#}_j\,\gamma^{I_j}_{q_j\dot{q}_j}\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot{q}_j I_{j+1}\ldots I_n}\quad :$$

- First, we contract $SO(8)_i$ 8v indices with $U_{I\,i}$ $(\gamma^I_{q\dot{p}}U_{I\,i}=2\bar{w}_{qAi}w^A_{\dot{p}i})$

$$\tilde{\mathcal{A}}_n(\{\rho^{\#}_{(i)},v^{-}_{\alpha q(i)};w_i,\bar{w}_i;\theta^{-}_{qi}\})=U_{I_1 1}\ldots U_{I_n n}\,\mathcal{A}_{I_1\ldots I_n}(\{\rho^{\#}_i,v^{-}_{\alpha qi};\theta^{-}_{qi}\})\,,$$

- we obtain the object which obeys

$$\bar{D}^{+(j)}_A\tilde{\mathcal{A}}_n(\{\rho^{\#}_{(i)},v^{-}_{\alpha q(i)};w_i,\bar{w}_i;\theta^{-}_{qi}\})=0\qquad\forall j=1,\ldots,n\,,$$

$$\bar{D}^{+(j)}_A=\bar{w}_{qAj}D^{+(j)}_q=\frac{\partial}{\partial\bar{\eta}^{-A}_j}+2\rho^{\#}_j\eta^{-}_{Aj}\,,\qquad\eta^{-}_{Aj}=\theta^{-}_{qj}\bar{w}_{qAj}=(\bar{\eta}^{-A}_j)^{*}\,.$$

- Our analytic 10D SYM superamplitude is related to this by

$$\mathcal{A}_n(\{\rho^{\#}_i,v^{-}_{\alpha qi};w_i,\bar{w}_i;\eta_{Ai}\})=e^{-2\sum_j\rho^{\#}_j\eta^{-}_{Bj}\bar{\eta}^{-B}_j}\,\tilde{\mathcal{A}}_n(\{\ldots,\bar{w}_i;\ \eta^{-}_{Ai}w^{A}_{qi}+\bar{\eta}^{-A}_i\bar{w}_{qAi}\})$$

Analytic superamplitude of 11D SUGRA

- The analytic superamplitudes of 11D SUGRA are constructed as

$$\mathcal{A}_n(\{\rho^{\#}_i, v^{-}_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\}) = U_{I_1 1} U_{J_1 1} \ldots U_{I_n n} U_{J_n n} \times$$
$$\times e^{-2\sum_j \rho^{\#}_j \eta^{-}_{Bj} \bar{\eta}^{-B}_j} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_j J_j)\ldots(I_n J_n)}(\{\rho^{\#}_i, v^{-}_{\alpha q i};\ \eta^{-}_{Ai} w^{A}_{qi} + \bar{\eta}^{-A}_i \bar{w}_{qAi}\}).$$

from the basic 11D superamplitude $\mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_j J_j)\ldots(I_n J_n)}$ obeying

$$D^{+}_{qj} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_j J_j)\ldots(I_n J_n)} = \rho^{\#}_j \gamma_{qp(I_j|} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_{j-1} J_{j-1})\,|J_j)p\,(I_{j+1} J_{j+1})\ldots(I_n J_n)}$$

- Notice that, despite the similarity of the superfield structure of analytic superamplitudes with ones of D=4 $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA
- the generalization of 4D results to 10D and 11D is not straightforward
- and is still to be elaborated; also some problems are to be solved.
- A potentially useful tool for this is Lorentz covariant counterpart of the light cone gauge, fixed on spinor frame variables found in [JHEP2018].
- Another interesting direction is to search for BCFW-type recurrent relations of these analytic superamplitudes.

Outline

Discussion

- One of the aims of this talk was to convince you that the D=10, 11 Lorentz harmonic approach and(or) spinor moving frame formalism [Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91, Bandos, Zheltukhin 91-95, Galperin, Howe, Stelle 93, Bandos, Nurmagambetov 96, Bandos, Sorokin, ..., Uvarov,...], which, in contrast to Newmen-Penrose diad and Penrose twistor formalism, work(s) with highly constrained set of spinors,
- is useful, besides the superembedding approach
  - [Bandos, Pasti, Sorokin, Tonin, Volkov 95, Bandos, Sorokin, Volkov 95, Howe, Sezgin 96, Howe, Sezgin, West 97, Bandos, Sorokin, Tonin 97, ...]

  also in the on-shell amplitude calculations.
- Of course, this approach is still at the initial stages of its development, and was not elaborated too intensively after 2018.
- Of related results I can mention that the polarized scattering equation [Geyer, Lipstein, Mason PRL14, Geyer, Mason 2019], a kind of square root of CHY scattering equation approach [Cachazo, He, Yuan, PRL 2014= arXiv:1307.2199], was obtained from 11D ambitwistor superstring in this frame in [JHEP 2019]).
- However, recent renewed interest to higher dimensional amplitudes [Herderschee + Maldacena 2023, Kallosh 2024] suggests to come back to attacking this problem.

Outlook

- Probably development of spinor moving frame based formalisms for amplitudes and superamplitudes of type IIB supergravity, an alternative to the existing study which used the natural complex structure of type IIB superspace, may help to develop both these lines.
- After better understanding and further elaborating the constrained and analytic superamplitude formalism for trees, it will be natural to search for their generalization for the case of loop (super)amplitudes.
- More speculatively sounds: to search for possible generalization for superstring superamplitudes (beyond the field theory), probably on the basis of superembedding approach, and
- ? to search for the generalizations for 11D superamplitudes beyond 11D SUGRA? (?M-theory amplitudes?)

THE END!

THANK YOU FOR YOUR ATTENTION!

Outline

It is convenient to introduce an auxiliary spinor frame $(v^{-}_{\alpha q}, v^{+}_{\alpha q})$ and associated vector frame $(u^{=}_{a}, u^{\#}_{a}, u^{I}_{a})$. Then

- any of the spinor and vector frames $(v^{-}_{\alpha q(i)}, v^{+}_{\alpha q(i)})$ and vector frame $(u^{=}_{a(i)}, u^{\#}_{a(i)}, u^{I}_{a(i)})$ 'attached' to one of the scattered particles are related to these by the Spin(1,D-1) Lorentz transformations
- but only $(D-2)$ of the parameters of this Lorentz transformation, $K_i^{=I}$ $(\approx \mathbb{S}^{D-2)})$, are not related to gauge symmetries which are used to define spinor frame(s)
- thus we can fix the gauge in which any spinor frame can be expressed through the auxiliary frame by

$$v^{-}_{\alpha q(i)} = v^{-}_{\alpha q} + \frac{1}{2} K_i^{=I} \gamma^{I}_{qp} v^{+}_{\alpha p} \, , \qquad v^{+}_{\alpha q(i)} = v^{+}_{\alpha q} \, .$$

- The frame vectors are related to the vectors of auxiliary frame by

$$u^{=}_{a(i)} = u^{=}_{a} + K_{(i)}^{=I} u^{I}_{a} + \frac{1}{4} (\vec{K}^{=}_{(i)})^2 u^{\#}_{a} \, ,$$

$$u^{I}_{a(i)} = u^{I}_{a} + \frac{1}{2} K_{(i)}^{=I} u^{\#}_{a} \, , \qquad u^{\#}_{a(i)} = u^{\#}_{a} \, .$$

- Gauge fixed form of the 3 point analytic superamplitude in 10D SYM

$$\mathcal{A}_3^{D=10\ SYM} = \frac{(\tilde{\rho}_1^{\#}\tilde{\rho}_2^{\#}\tilde{\rho}_3^{\#})^2 e^{-2i(\beta_1+\beta_2+\beta_3)}}{\langle 12]\ \langle 23]\ \langle 31]}\ \delta^4\left(\frac{\tilde{q}^-_{A[1]} - \tilde{q}^-_{A[2]}}{\tilde{\rho}_3^{\#}}\right).$$

where $\boxed{\langle ij] := \frac{1}{4}\rho_j^{\#}(v_{qi}^{+\alpha} v^-_{\alpha pj})\ \overline{\Psi}_{q\dot{p}i}\ (v_{\dot{p}i}^{-\beta} v^-_{\beta pj})}$

- and $q^-_{A[i]} = \bar{w}_{\dot{q}Ai} v_{\dot{q}i}^{-\alpha} \sum_{j=1}^{3} \rho_i^{\#} v^-_{\alpha pj}\theta^-_{pi}$ is analytic supermomentum,

$\frac{\partial}{\partial\bar{\eta}_j^{-B}}\, q^-_{A[i]} = 0 \qquad \forall \quad i,j = 1,2,3$

- and $e^{2i\beta_i}$ is defined by $U_{Ii} = e^{2i\beta_i} U_J \mathcal{O}_i^{JI}$, $\bar{U}_{Ii} = e^{-2i\beta_i}\bar{U}_J\mathcal{O}_i^{JI}$, or

$$\bar{w}_{qA\,i} = \mathcal{O}_{qp\,i}\bar{w}_{pB}\ e^{i\beta_i}\,\mathcal{U}^{\dagger\,B}_{A\,i}\ , \qquad w_{q\,i}^{\ A} = \mathcal{O}_{qp\,i} w_p^{\ A}\ e^{-i\beta_i}\mathcal{U}_{B\,i}^{A}\ ,$$
$$\mathcal{U}_{B\,i}^{A} \in SO(D-4) \subset SU(\mathcal{N})\ ,$$

- while $v^-_{\alpha qi} = e^{-\alpha_i}\mathcal{O}_{iqp}\left(v^-_{\alpha p} + \frac{1}{2}K_i^{=I}\gamma^I_{p\dot{q}} v^+_{\alpha\dot{q}}\right)$, $\mathcal{O}_{iqp} \subset SO(D-2)$,
- and $\tilde{q}^-_{A[i]} - \tilde{q}^-_{A[j]} = e^{\alpha_i + i\beta_i}\mathcal{U}_{Ai}^{B}\left(q^-_{B[i]} - v_{Bi}^{-\alpha}\bar{v}^{+C}_{\alpha j} q^-_{C[j]}\right)$.

- Gauge fixed form of the 3 point analytic superamplitude of 11D SG is

$$\mathcal{A}_3^{D=11\ SUGRA} = \frac{(\tilde{\rho}_1^{\#}\tilde{\rho}_2^{\#}\tilde{\rho}_3^{\#})^4 e^{-2i(\beta_1+\beta_2+\beta_3)}}{\langle 12]^2\ \langle 23]^2\ \langle 31]^2}\ \delta^8\left(\frac{\tilde{q}^-_{A[1]} - \tilde{q}^-_{A[2]}}{\tilde{\rho}_3^{\#}}\right)$$

- where

$$\delta^8\left(\frac{\tilde{q}^-_{A[1]} - \tilde{q}^-_{A[2]}}{\tilde{\rho}_3^{\#}}\right) \equiv \frac{1}{8!}\,\epsilon^{A_1\ldots A_8}\left(\tilde{q}^-_{A_1[1]} - \tilde{q}^-_{A_1[2]}\right)\ldots\left(\tilde{q}^-_{A_8[1]} - \tilde{q}^-_{A_8[2]}\right).$$

- Let us introduce a complex spinor frame

$$v^{-}_{\alpha A} := v^{-}_{\alpha q}\bar{w}_{qB}, \qquad v^{+}_{\alpha A} := v^{+}_{\alpha\dot{p}}\bar{w}_{\dot{p}B} \;, \qquad \bar{v}^{-A}_{\alpha} := v^{-}_{\alpha p}w_{p}{}^{A} \;, \qquad \bar{v}^{+A}_{\alpha} := v^{+}_{\alpha\dot{p}}w_{\dot{p}}{}^{A} \;.$$

- The BCFW deformation of this spinor frame and of the fermionic variables read

$$\widehat{v^{-}_{\alpha A(n)}} = v^{-}_{\alpha A(n)} + z\, v^{-}_{\alpha A(1)} \sqrt{\rho^{\#}_{1}/\rho^{\#}_{n}} \;, \qquad \widehat{\bar{v}^{A-}_{\alpha(n)}} = \bar{v}^{A-}_{\alpha(n)} \;,$$

$$\widehat{v^{-}_{\alpha A(1)}} = v^{-}_{\alpha A(1)} \;, \qquad \widehat{\bar{v}^{A-}_{\alpha(1)}} = \bar{v}^{A-}_{\alpha(1)} - z\, \bar{v}^{A-}_{\alpha(n)} \sqrt{\rho^{\#}_{n}/\rho^{\#}_{1}}$$

and

$$\widehat{\eta^{-}_{A\,n}} = \eta^{-}_{A\,n} + z\, \eta^{-}_{A\,1} \sqrt{\rho^{\#}_{1}/\rho^{\#}_{n}} \;, \qquad \widehat{\eta^{-}_{A\,1}} = \eta^{-}_{A\,1}$$