On the d and M Conjecture

by John Lewis

Abstract

Let $n \geq 2$ be a positive integer, $x = (x_1, x_2, \ldots, x_n)$ a point in Euclidean n space, $\mathbb{R}^n$, and let $|x|$ denote the norm of x. Put $B(x,r) = \{x : |x| < r\}$ when $r > 0$. For fixed $n \geq 2$, let μ be a positive Borel measure on $\mathbb{S}^{n-1} = \{x : |x| = 1\}$ with $\mu(\mathbb{S}^{n-1}) = 1$. Fix $d \leq M$, and let $\mathcal{F}_d^M$ denote the family of potentials p with $d \leq p \leq M$ satisfying

$$(a) \quad p(x) = \int_{\mathbb{S}^{n-1}} |x-y|^{2-n} d\mu(y), x \in \mathbb{R}^n, \text{ when } n > 2,$$

$$(b) \quad p(x) = 2\int_{\mathbb{S}^{n-1}} \log \tfrac{1}{|x-y|} \, d\mu(y), x \in \mathbb{R}^2.$$

Let $\mathcal{H}^{n-1}$ denote surface area on $\mathcal{S}^{n-1}$ and let Φ be an increasing convex function on $\mathbb{R}$. In this talk we discuss progress on the following conjecture:

Conjecture: If $\mathcal{F}_d^M \neq \emptyset$, then for $0 < r < \infty$,

$$\int_{\mathbb{S}^{n-1}} \Phi(p(ry)) d\mathcal{H}^{n-1} y \leq \int_{\mathbb{S}^{n-1}} \Phi(P(ry)) d\mathcal{H}^{n-1} y$$

where $P = P(\cdot, d, M) \in \mathcal{F}_d^M$ is unique up to a rotation of $\mathbb{S}^{n-1}$ (independently of Φ) and defined as follows: There exist (also unique) α, β with $-1 \leq \beta < \alpha \leq 1$, and

$$\begin{array}{ll} (a) & P(\cdot, d, M) \equiv M \text{ on } E_1 = \{x \in \mathcal{S}^{n-1} : \alpha \leq x_1 \leq 1\} \\ (b) & P(\cdot, d, M) \equiv d \text{ on } E_2 = \{x \in \mathbb{S}^{n-1} : -1 \leq x_1 \leq \beta\} \\ (c) & P(\cdot, d, M) \text{ is harmonic in } \mathbb{R}^n \setminus (E_1 \cup E_2). \end{array}$$