Extension of the Tricomi problem for a loaded parabolic-hyperbolic type equation

Umida Baltaeva

Khorezm Mamun Academy, Urgench State University, Uzbekistan

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Outline

The theory of mixed type equations is one of the principal parts of the general theory of partial differential equations. The interest for these kinds of equations arises intensively due to both theoretical and practical uses of their applications. The first fundamental results in this direction were obtained in 1923 by F. Tricomi. The works of S.Gellerstedt, M.A.Lavrent'ev, A.V.Bitsadze, F.I.Frankl, M.Protter, M.S.Salakhitdinov and T.D.Djuraev have had a great impact in this theory, where outstanding theoretical results were obtained and pointed out important practical values.

The necessity of the consideration of the parabolic-hyperbolic type equation was specified for the first time in 1956 by I.M.Gel' fand. He gave an example connected to the movement of the gas in a channel surrounded by a porous environment. The movement of the gas inside the channel was described by the equation, outside by the diffusion equation.

In the recent years, in connection with intensive research on problems of optimal control of the agro economical system, mathematical biology, long-term forecasting and regulating the level of ground waters and soil moisture, it has become necessary to investigate a new class of equations called "loaded equations". Such equations were investigated for the first time by A.Knezer(1914), L.Lichtenstein(1931), N.N.Nazarova(1937). This terminology has been introduced by A.M.Nakhushev(1976), where the most general definition of a loaded equation is given and various loaded equations are classified in detail, e.g., loaded differential, integral, integro-differential, functional equations etc., and numerous applications are described.

1. - Mixed equations of parabolic-hyperbolic, elliptic-hyperbolic types:
- gas dynamics,
- electromagnetic fields,
- magneto hydrodynamics,

2 Many problems in mathematical physics and mathematical biology,
- linearization of nonlinear equations,
- investigation of inverse problems,
- long-term forecasting and regulating the level of ground waters and soil moisture
- problems optimal control agro ecosystems

reduced to boundary-value problems for the loaded partial differential equations.

Definition (Loaded equation)

An equation

$$Au(x) = f(x) \tag{0.1}$$

is called loaded in n-dimensional Euclidean domain Ω, if (part of) the operator A depends on the restriction of the unknown function $u(x)$ defined on the closed subset $\overline{\Omega}$ of measure strictly less than n.

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Class of researchers confirming the actuality of loaded differential equations is a phenomena in complex evolutionary systems with memory essentially depending on the prehistory of this system and these phenomena are described by the following loaded integro-differential equations of elliptic type [1]

$$\Delta_x u(x,t) + \int\limits_0^t \sum_{j=1}^n k_j(t,\tau)\frac{\partial^2 u(x,\tau)}{\partial x_j^2}d\tau = 0,$$

parabolic type

$$\frac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t) + \int\limits_0^t k_0(t,\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau,$$

where $\Delta_x -$ Laplace operators at $x = (x_1, x_2, x_3)$, $k_0(t,\tau)$, $k_j(x,\tau)$ are given real-valued functions, $j = 1, 2, \ldots, n$.

The following loaded partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = mc_0^2 \frac{\partial^2}{\partial x^2} \int_{-\infty}^{t} \exp\left(-\frac{t-\eta}{\tau_0}\right) \frac{\partial u(x,\eta)}{\partial \eta} d\eta,$$

is found in the theory of wave propagation in dispersive environments [1], where $u(x,t)$ is the density, c_0 is the speed, m is the mass, and τ_0 is the time of relaxation.
The following loaded equation of the first order [3]

$$\frac{\partial \varphi}{\partial t} + y\frac{\partial \varphi}{\partial x} = (1-\rho)\varphi \int_{\alpha}^{\beta} (y-\eta)\varphi(x,t,\eta)d\eta - \frac{1}{\tau_0}\left[\varphi - G(y)\int_{\alpha}^{\beta} \varphi(x,t,\eta)d\eta\right.$$

is the equation modeling the movement of vehicles on the highway, where $\varphi(x,t,y)$ is the density of the car in the points $x \in [a,b] \subset R$ having a speed, m is the mass, τ_0 is the time of relaxation, and p and G are the given values; moreover, the

As know from [1] non-local problem for the integro - differential equations are directly relevant to the problem for such as calling "loaded differential", "loaded integro-differential" equations.
At the Mc Kendrick and Von Foerster equation

$$\frac{\partial u(a,t)}{\partial a}+\frac{\partial u(a,t)}{\partial t}+c(a,t)u(a,t)=0,$$

close population sufficiently well described. $u(0,t)=\int\limits_0^l k(x,t)u(x,t)dx,\ 0\le t\le T,\ u(a,0)=\tau(0), 0\le a\le 1.$
In the case when from population in each time t, removed singular ages $a_1,\ a_2,\ ...a_n$, equation can be write in the form of

$$\frac{\partial u(a,t)}{\partial a}+\frac{\partial u(a,t)}{\partial t}+c(a,t)u(a,t)+\sum_{i=1}^{n}c_i(a,t)u(a_i,t)=0,$$

which is relative to the "loaded differential equation" in the form [1]

The following equation arises in problems of some mathematical models, in problems of particle transfer in the plane-parallel geometry

$$\frac{1}{c}\frac{\partial\omega(z)}{\partial z_3} + z_2\frac{\partial\omega(z)}{\partial z_1} + \sigma(z_1)\omega(z) = \frac{\sigma_3(z_1)}{2}\int\limits_{-1}^{1}\omega(z)dz_2 + f(z),$$

where $\omega(z) = \omega(z_1, z_2, z_3)$ is the density grain in point z_1 at time $z_3 \geq 0$ flying with speed c under angle θ, $\cos\theta = z_2$, to straight line $z_3, z_2 = 0$. Moreover the functions $\sigma(z_1)$, $\sigma_8(z_1)$ and $f(z)$ are given. Also the loaded equation will be the stationary unispeed equation of transport (we can see [3])

$$\frac{1}{\alpha(z)}\frac{\partial\varphi(y,z)}{\partial z_j} + \varphi(y,z) = \frac{\lambda}{4\pi}\int\limits_{|\xi|=1}\theta(z,y,\xi)\varphi(\xi,z)d\xi + F(y,z)$$

in the phase domain $\{(y, z) : |y| = \sqrt{y_1^2 + y_2^2 + y_3^2} = 1, z \in \Omega\}$

Let us consider the following analogue of the Darboux problem for the loaded equation of hyperbolic type

$$\frac{\partial}{\partial x}\left(u_{xx}-u_{yy}-\lambda u\right)-\mu\sum_{i=1}^{n}a_i(x)D_{0x}^{\alpha_i}u(x,0)=0, \qquad (1)$$

where $D_{ax}^{\alpha}\varphi(x)$ is integro-differential operator (in the sense of Riemann-Liouville):

$$D_{ax}^{\alpha}\varphi(x)=\begin{cases}\frac{sign(x-a)}{\Gamma(-\alpha)}\int\limits_a^x\frac{\varphi(t)dt}{|x-t|^{1+\alpha}}, & \alpha<0,\\ \varphi(x), & \alpha=0,\\ sign(x-a)\frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}}D_{ax}^{\alpha-[\alpha]-1}\varphi(x), & \alpha>0.\end{cases} \qquad (2)$$

Assume, that $\alpha_n<\alpha_{n-1}<...<\alpha_1=\alpha<1$ and coefficients $a_i=a_i(x)\in C^1[0,1]$, $\lambda,\mu-$ are given real parameters.

1. Let $D \subset R^2 -$ be a domain, bounded at $y < 0$ by the characteristics AC, BC of equation (1) and the segment AB of the axis $y = 0$.

Darboux problem. Find a solution $u(x, y)$ to equation (1), which is regular in the domain D, continuous in $\bar{D}$, and has continuous derivatives u_x, u_y, up to $AB \cup AC$, and satisfies the boundary value conditions

$$u_y(x, y)|_{AB} = \nu(x), \ 0 \leq x < 1, \tag{3}$$

$$u(x, y)|_{AC} = \psi_1(x), \quad \left.\frac{\partial u(x, y)}{\partial n}\right|_{AC} = \psi_2(x), \ 0 \leq x \leq \frac{1}{2}, \tag{4}$$

where $n -$ is the inner normal, $\nu(x), \psi_1(x), \psi_2(x) \ -$ are real-valued functions.

Theorem (General representation of solutions)

Any regular solution of equation (1) is represented in the form

$$u(x,y) = z(x,y) + w(x), \tag{5}$$

where $z(x,y)$ *is a solution of the equation*

$$\frac{\partial}{\partial x}(z_{xx} - z_{yy} - \lambda z) = 0, \tag{6}$$

and $w(x)$ *is the solution of the following ordinary differential equation*

$$w'''(x) - \lambda w'(x) - \mu \sum_{i=1}^{n} a_i D_{0x}^{\alpha_i} w(x) = \mu \sum_{i=1}^{n} a_i D_{0x}^{\alpha_i} z(x,0). \tag{7}$$

Invoking that the function $ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x} + c$ satisfies equation (6), we can assume without loss of generality that

$$w(0) = w'(0) = w''(0) = 0. \tag{8}$$

when studying the Darboux problem.

The solution to the Cauchy problem for equation (7) with the conditions (8) has the form

$$w(x) = \int_0^x P(x,t) \sum_{i=1}^{n} a_i(t) D_{0x}^{\alpha_i} z(t,0)dt\ . \tag{9}$$

where

$$P(x,t) = \frac{\mu}{\lambda}\{\cos(\sqrt{\lambda}(x-t)) - 1 + \mu \int_t^x \cos(\sqrt{\lambda}(s-t)) - 1)R(x,s;\mu)c \tag{10}$$

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(10)

If $a_i(x) = 1, \alpha_i(x) = 0$, and $\lambda = -3\left(\mu/2\right)^{\frac{2}{3}}$ has the form

$$w(x) = \int_0^x T(x,t)z(t,0)dt,$$

where

$$T(x,t) = \frac{2}{9}\left(\frac{\mu}{2}\right)^{\frac{1}{3}} e^{\sqrt[3]{\frac{\mu}{2}}(x-t)}\left(e^{\sqrt[3]{\frac{\mu}{2}}(x-t)} - 3\left(\frac{\mu}{2}\right)^{\frac{1}{3}}(x-t) - 1\right).$$

By virtue of the representation (7), the D problem is reduced to the D* problem of finding a solution $z(x, y)$ of equation (6), which is regular in the domain with the conditions

$$z_y(x, y)|_{AB} = \nu(x),\ 0 < x < 1$$

$$z|_{AC} = \psi_1(x) - w(x),\ \frac{\partial z}{\partial n}\bigg|_{AC} = \psi_2(x) - \frac{1}{\sqrt{2}} w'(x),\ \ 0 \le x \le \frac{1}{2},$$

where $\omega(x)$ is defined by (9)

Let $\Omega_1 \subset R^2$ be a bounded domain enclosed by the segments AB, BB_0, AA_0, A_0B_0 of straight lines
$y = 0, \; x = 1, \; x = 0, \; y = h$, respectively when $y > 0$. Ω_2 is a characteristic triangle bounded by the segment AB the axis OX and two characteristics

$$AC : x + y = 0, \quad BC : x - y = 1$$

of the wave equation for $y < 0$.
We introduce the following notation:

$$I = \{(x, y) : 0 < x < 1, \; y = 0\}, \quad \Omega = \Omega_1 \cup \Omega_2 \cup I,$$

$$J_1 = \{(x, y) : 0 < y < h, \; x = 0\}, \quad J_2 = \{(x, y) : 0 < y < h, \; x = 1\}.$$

We consider the following linear loaded equation of mixed parabolic-hyperbolic type

$$u_{xx}-\frac{1-sgny}{2}u_{yy}-\frac{1+sgny}{2}u_y-Mu(\Theta(x),0)=0, \qquad (1)$$

where $Mu(\Theta(x),0)=\mu_1 D_{0x}^{\alpha_i}u(x,0)$ in Ω_1, $Mu(\Theta(x),0)=\mu_2 D_{0\xi}^{\beta_i}u(\xi,0)$, $\xi=x-y$, in Ω_2, μ_1, μ_2 are given real parameters, $D_{ax}^{\alpha_i}$ ($D_{a\xi}^{\beta_i}$) are the Riemann-Liouville fractional integro-differential operator of order α_i (β_i).

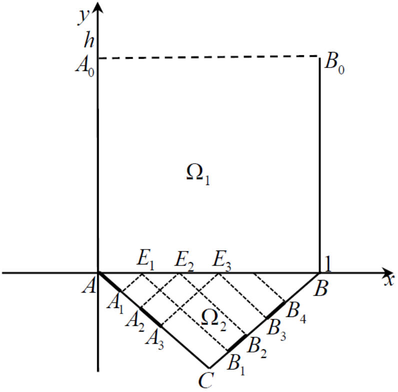

Figure 1

We denote by $E_k(\epsilon_k, 0)(k = 1, ...n)$, $0 < \epsilon_1 < ... < \epsilon_n < 1$ given points of the segment AB, and

$$I_k = \{(x, y) : \epsilon_k < x < \epsilon_{k+1}, \quad y = 0\}, k = 0, 1, ..., n.$$

Problem T_1. Find a function $u(x,y)$, satisfying the following conditions:

1) $u(x,y) \in C(\overline{\Omega})$, has continuous derivatives and the derivatives of the first order are continuously glued together over all points of the segment AB, except, possibly, points $A,\ E_1, ..., E_n,\ B$;

2) $u(x,y)$ is a regular solution to equation (1) in the domain Ω;

3) takes the given values

$$u(x,y)|_{x=0} = \varphi_1(y), \quad u(x,y)|_{x=1} = \varphi_2(y), \quad 0 \le y \le h, \qquad (2)$$

$$u(x,y)|_{A_k A_{k+1}} = \psi_k(x), \text{ for even } k; \qquad (3)$$

$$u(x,y)|_{B_k B_{k+1}} = \psi_k(x), \text{ for odd } k; \qquad (4)$$

where $\varphi_1(y),\ \varphi_2(y), \psi_k(x),\ (k = 0, 1, ..., n)$ are the given sufficiently smooth functions.

From condition 1) in problem T_1 it follows that

$$u(x,+0) = u(x,-0) = \tau(x), \quad (x,0) \in \bar{I}_k, \tag{5}$$

$$u_y(x,+0) = u_y(x,-0) = \nu(x), \quad (x,0) \in I_k, \tag{6}$$

where $\tau(x)$ and $\nu(x)$ are smooth functions.

Let n be an even number. Asgeirsson principle:

$$u(\epsilon_{2k},0)=\tau(\epsilon_{2k})=\psi_{2k}\left(\frac{\epsilon_{2k}}{2}\right)+\psi_{2k-1}\left(\frac{\epsilon_{2k}+1}{2}\right)-$$

$$-\psi_n\left(\frac{1}{2}\right)=a_{2k},\ k=1,2,...,\frac{n}{2}. \tag{7}$$

$$u(\epsilon_{2k+1},0)=\tau(\epsilon_{2k+1})=\psi_{2k}\left(\frac{\epsilon_{2k+1}}{2}\right)+$$

$$+\psi_{2k+1}\left(\frac{\epsilon_{2k+1}+1}{2}\right)-\psi_n\left(\frac{1}{2}\right)=a_{2k+1},\ k=0,1,...,\frac{n-2}{2}. \tag{8}$$

The solution to the Cauchy problem for equation (1) in Ω_2, with the conditions (5) and (6), has the form:

$$u(x,y)=\frac{1}{2}\left[\tau(x+y)+\tau(x-y)\right]-\frac{1}{2}\int\limits_{x+y}^{x-y}\nu(\xi)\,d\xi+$$

$$+\frac{\mu_2}{4}\int\limits_{x+y}^{x-y}(x-y-\xi)\ D_{0\xi}^{\beta_i}\tau(\xi)d\xi. \tag{9}$$

From (9) taking into account (3), we obtain the following functional relation from the domain Ω_2 to I_{2k}:

$$\tau'(2x) - \nu(2x) + \frac{\mu_2}{2} \int\limits_0^{2x} D_{0\xi}^{\beta_i} \tau(\xi) d\xi = \psi'_{2k}(x),$$

i.e.

$$\tau'(x) - \nu(x) + \frac{\mu_2}{2} \int\limits_0^{x} D_{0\xi}^{\beta_i} \tau(\xi) d\xi = \psi'_{2k}\left(\frac{x}{2}\right), \; x \in I_{2k}, \; k = 0, 1, ..., \frac{n}{2}. \tag{10}$$

Similarly, applying (9) to (4), we get the following functional relation from the domain Ω_2 to I_{2k-1} at $k = 1, ..., n/2.$:

$$\tau'(x) + \nu(x) + \frac{\mu_2}{2}(1-x) D_{0x}^{\beta_i} \tau(x) = \psi_{2k-1}\left(\frac{x+1}{2}\right), \; x \in I_{2k-1}.$$

Now, passing to the limit as $y \to +0$ in equation (1), with taking into account 1) condition of the problem T_1, (5) and (6), we obtain the integro-differential relation between $\tau(x)$ and $\nu(x)$, transferred from the domain Ω_1 to I_k:

$$\tau^{''}(x) - \nu(x) - \mu_1 D_{0x}^{\alpha_i} \tau(x) = 0. \qquad (12)$$

Theorem

If the functions $u(x, y) \in C(\overline{\Omega}) \cap C^{1,2}_{x,y}(\Omega_1) \cap C^2(\Omega_2)$, *and* $u_x,\ u_y$ *glue together over all points of the segment AB, except, possibly, points* A, $E_1, ..., E_n$, B, *and where can only have discontinuities of the first kind, and satisfy conditions*

$$\alpha_i < 1,\ \ \beta_i < 1,\ \mu_1,\ \mu_2 \geq 0,$$

then the solution $u(x, y)$ *of problem* T_1 *is unique in the domain* Ω.

Theorem

Let $\varphi_k(y) \in C(\overline{J_k}) \cap C^1(J_k)$, $\psi_k(x) \in C(\overline{I_k}) \cap C^1(I_k)$, $\varphi_1(0) = \psi_0(0)$ *and, moreover, if n is an odd number, assume that*

$$\lim_{x \to +0} u_y(x,0) = \varphi_1'(0), \tag{13}$$

then there exists a unique solution to the problem T_1 *in the domain* Ω*.*

Eliminating from the system of equations (10), (12), and (11), (12) the functions $\nu(x)$, we get the following problems

$$\tau''(x) - \tau'(x) - \mu_1 D_{0x}^{\alpha_i}\tau(x) + \frac{\mu_2}{2}\int\limits_0^x D_{0t}^{\beta_i}\tau(t)dt = -\psi'_{2k}\left(\frac{x}{2}\right),$$

$$\tau(\varepsilon_{2k}) = a_{2k},\ \ \tau(\varepsilon_{2k+1}) = a_{2k+1},\ \ k = 0, 1, ..., n/2, \tag{14}$$

$$\tau''(x) + \tau'(x) - \mu_1 D_{0x}^{\alpha_i} \tau(x) - \frac{\mu_2}{2}(1-x) D_{0x}^{\beta_i} \tau(x) = \psi_{2k-1}\left(\frac{x+1}{2}\right),$$

$$\tau(\varepsilon_{2k-1}) = a_{2k-1}, \ \ \tau(\varepsilon_{2k}) = a_{2k}, \ k = 1, 2, ..., n/2, \tag{15}$$

where

$$\tau(0) = \varphi_1(0) = \psi_0(0), \ \tau(1) = \varphi_2(0).$$

From here, by considering the properties of the integral operators for $\alpha_i < 0$ and $\beta_i < 0$ we get the following integro-differential equation

$$\tau''(x) - \tau'(x) - \int\limits_0^x K_1(x,t)\tau(t)dt = -\psi'_{2k}\left(\frac{x}{2}\right), \qquad (16)$$

$$\tau''(x) + \tau'(x) - \int\limits_0^x \widehat{K}_1(x,t)\tau(t)dt = \psi'_{2k-1}\left(\frac{x+1}{2}\right), \qquad (17)$$

where

$$K_1(x,t) = \frac{\mu_1}{\Gamma(-\alpha_i)(x-t)^{\alpha_i+1}} + \frac{\mu_2}{2\Gamma(1-\beta_i)(x-t)^{\beta_i}}, \quad \alpha_i < 0, \beta_i < 0,$$

$$\widehat{K}_1(x,t) = \frac{\mu_1}{\Gamma(-\alpha_i)(x-t)^{\alpha_i+1}} + \frac{\mu_2(1-x)}{2\Gamma(-\beta_i)(x-t)^{\beta_i+1}}, \quad \alpha_i < 0, \beta_i < 0.$$

From (14) and (15), by considering the properties of the integral operators for $0 < \alpha_i < 1$ and $0 < \beta_i < 1$ we get the following integro-differential equation

$$\tau''(x) - \tau'(x) - \int\limits_0^x K_1(x,t)\tau'(t)dt = -\psi'_{2k}\left(\frac{x}{2}\right) - \\ -\frac{\mu_1 x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\varphi_1(0) + \frac{\mu_2 x^{1-\beta_i}}{2\Gamma(2-\beta_i)}\varphi_1(0), \quad (18)$$

$$\tau''(x) + \tau'(x) + \int\limits_0^x \widehat{K}_1(x,t)\tau'(t)dt = \psi'_{2k}\left(\frac{x+1}{2}\right) - \\ -\frac{\mu_1 x^{-\alpha_i}}{\Gamma(1-\alpha_i)}\varphi_1(0) - \frac{\mu_2 (1-x) x^{-\beta_i}}{2\Gamma(1-\beta_i)}\varphi_1(0), \quad (19)$$

where $K_1(x,t)$ and $\widehat{K}_1(x,t)$ are defined as above

At first, we investigate the integro-differential equation (16) with the conditions of the problem (14). To this end, we introduce the notation

$$F_1(x) = -\psi'_{2k}\left(\frac{x}{2}\right) + \int\limits_0^x K_1(x,t)\tau(t)dt, \tag{20}$$

so that equation (4.4) can be written in the form

$$\tau''(x) - \tau'(x) = -F_1(x).$$

Hence, if we solved the latter equation, we have

$$\tau(x) = \int\limits_{\epsilon_{2k}}^x \left(e^{x-t} - 1\right) F_1(t)dt + c_1 + c_2 e^x, \tag{21}$$

Hence, with the preceding notation and after a few transformations, we have:

$$\tau(x) - \int\limits_{0}^{\epsilon_{2k+1}} K_{11}(x,t)\tau(t)dt = \int\limits_{\epsilon_{2k}}^{x} \left(e^{x-t} - 1\right) \psi'_{2k}\left(\frac{t}{2}\right) dt +$$

$$+ \int\limits_{\epsilon_{2k}}^{\epsilon_{2k+1}} \frac{(e^{x} - e^{\epsilon_{2k}})(e^{\epsilon_{2k+1}-t} - 1)}{e^{\epsilon_{2}k} - e^{\epsilon_{2k+1}}} \psi'_{2k}\left(\frac{t}{2}\right) dt +$$

$$+ \frac{a_{2k}(e^{x} - e^{\epsilon_{2k+1}}) - a_{2k+1}(e^{x} - e^{\epsilon_{2k}})}{e^{\epsilon_{2}k} - e^{\epsilon_{2k+1}}} \quad \epsilon_{2k} \leq x \leq \epsilon_{2k+1}, \qquad (22)$$

where

$$K_{11}(x,t) = K_1^*(x,t;\epsilon_{2k}) + \frac{e^x - e^{\epsilon_{2k}}}{e^{\epsilon_{2k}} - e^{\epsilon_{2k+1}}} K_1^*(\epsilon_{2k+1}, t; \epsilon_{2k}), \ 0 \le t \le \epsilon_{2k},$$

$$K_{11}(x,t) = K_1^*(x,t;t) + \frac{e^x - e^{\epsilon_{2k}}}{e^{\epsilon_{2k}} - e^{\epsilon_{2k+1}}} K_1^*(\epsilon_{2k+1}, t; t), \ \epsilon_{2k} \le t \le x,$$

$$K_{11}(x,t) = \frac{e^x - e^{\epsilon_{2k}}}{e^{\epsilon_{2k}} - e^{\epsilon_{2k+1}}} K_1^*(\epsilon_{2k+1}, t; t), \ x \le t \le \epsilon_{2k+1},$$

$$K_1^*(x,t;\xi) = \int\limits_{\xi}^{x} \left(e^{x-s} - 1\right) K_1(s,t)ds.$$

Thus, it follows from the uniqueness of the problem, and from the theory of Fredholm integral equations of the second kind, that equation (22), taking account of conditions of Theorem 2, has a unique solution, which has the form:

$$\tau(x) = \Phi(x) + \int_0^{\epsilon_{2k+1}} R_{11}(x,t)\Phi(t)dt, \qquad (23)$$

where $R_{11}(x,t)$ is the resolvent of the kernel $K_{11}(x,t)$, and its solution $\tau(x)$ is continuous in $\bar{I}_k$, has continuous derivatives and the derivatives of the first order are continuously glued together over all points of the segment I_k, except, possibly, points $A, E_1, ..., E_n, B$, in which $\tau'(x)$ can only have discontinuities of the first kind.

Similarly, we have

$$\tau(x) - \int_{0}^{\epsilon_{2k}} \overline{K}_{11}(x,t)\tau(t)dt = - \int_{\epsilon_{2k-1}}^{x} (e^{x-t} - 1)\psi'_{2k-1}\left(\frac{t}{2}\right) dt +$$

$$+ \int_{\epsilon_{2k-1}}^{\epsilon_{2k}} \frac{(e^{-x} - e^{-\epsilon_{2k-1}})(e^{-(\epsilon_{2k}-t)} - 1)}{e^{-\epsilon_{2k-1}} - e^{-\epsilon_{2k}}}\psi'_{2k-1}\left(\frac{t}{2}\right) dt +$$

$$+ \frac{a_{2k-1}(e^{-x} - e^{-\epsilon_{2k}}) - a_{2k}(e^{-x} - e^{-\epsilon_{2k-1}})}{e^{-\epsilon_{2k-1}} - e^{-\epsilon_{2k}}}, \quad \epsilon_{2k-1} \le x \le \epsilon_{2k}.$$

Analogously, taking into account the uniqueness of the problem T_1 and the theory of integral equation, solving the integral equation we have:

$$\tau(x) = \overline{\Phi}(x) + \int_0^{\epsilon_{2k+1}} \overline{R}_{11}(x,t)\overline{\Phi}(t)dt, \qquad (24)$$

where $\overline{R}_{11}(x,t)$ is the resolvent of the kernel $\overline{K}_{11}(x,t)$, and the functions $\overline{\Phi}(x)$, $\overline{K}_{11}(x,t)$ depend on the properties of these functions established as above.

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Thank you for your attention!