THE GENERALIZED SYLVESTER'S AND ORCHARD PROBLEMS VIA DISCRIMINANTAL ARRANGEMENT

Pragnya Das

Hokkaido University

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Objective

- We aim at elucidating the connection between the generalised Sylvester's and Orchard Problem and the combinatorics of discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$.
- We answer the above question for a special case of arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$.
- An arrangement of lines is a finite collection of lines in a plane. The point where r lines intersect is called a multiplicity of r intersection.

The generalised Sylvester's Problem

When posed in its dual form leads to the question: given an arrangement of n lines in $\mathbb{C}^2$ what is the minimum number of multiplicity 2 intersections.

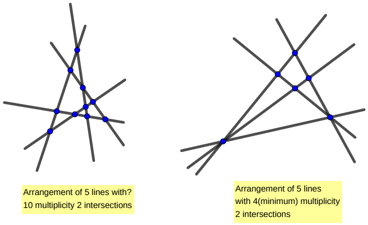

Figure 1: Examples of arrangement with 5 lines

The generalised Orchard Problem

When posed in its dual form leads to the question: given an arrangement of n lines in $\mathbb{C}^2$ what is the maximum number of multiplicity 3 intersections.

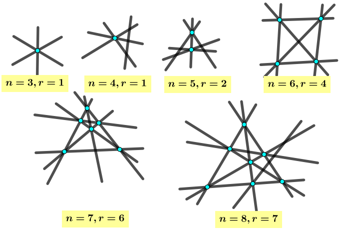

Figure 2: Examples of Orchard problem with n lines and multiplicity 3 intersections

Background

- Pappus's configuration is an arrangement of 6 lines with 3-collinearity conditions.
- Pappus's configuration with 3-collinearity conditions is denoted by $\mathcal{P}_\infty$.
- Pappus's configuration with 4-collinearity conditions is denoted by $\mathcal{P}^c_\infty$.

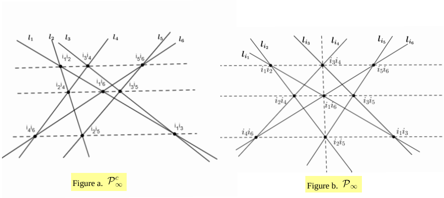

Figure 3: Pappus's configurations

Geometrical Approach

In our problem we consider a Pappus's configuration where the three classical collinearities are concurrent.
Six new lines are added to the 6 lines in Pappus's configuration in the following way to get the arrangement of 12 lines :

1. lines l'_1, l'_2, l'_3 are the three concurrent lines corresponding to the three Pappus's collinearities;
2. lines l'_4, l'_5, l'_6 are added so that each one of them contains exactly two different multiplicity 2 intersection of $\mathcal{P}^c_\infty$ (resp. $\mathcal{P}_\infty$) and that each multiplicity 2 intersection is contained in only one line $l'_i, i = 1, \ldots, 6$.

Arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$

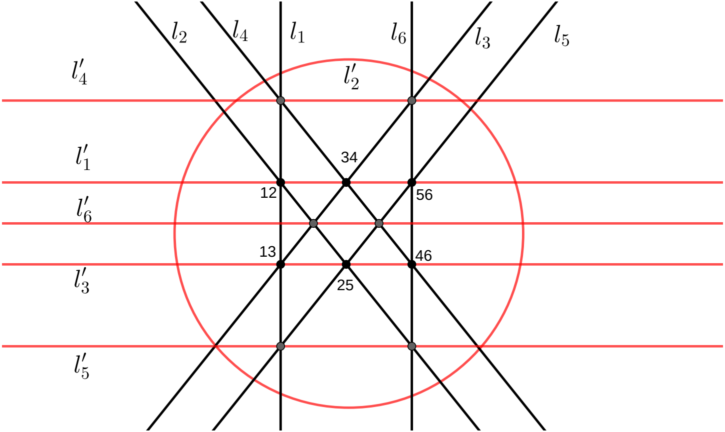

Figure 4: Arrangement of 12 lines with 6 multiplicity 2 intersections in $\mathbb{P}^2\mathbb{R}$ where the black lines depict the Pappus's configuration.

Arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$

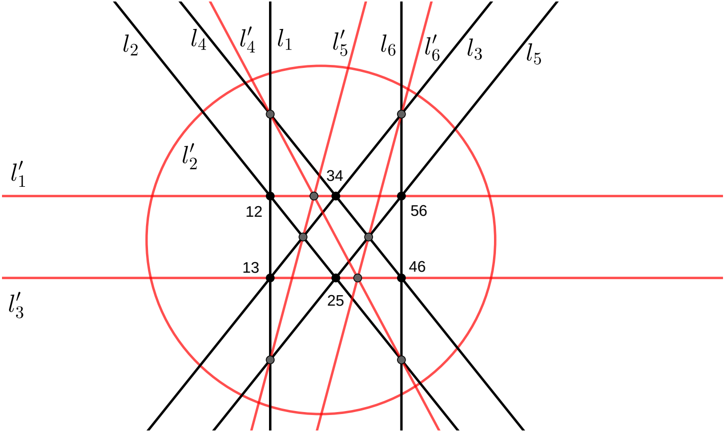

Figure 5: Arrangement of 12 lines with 19 multiplicity 3 intersection in $\mathbb{P}^2\mathbb{R}$ where the black lines depict the Pappus's configuration.

Discriminantal Arrangement

- The discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is an arrangement of hyperplanes, constructed from a generic arrangement $\mathcal{A}$, generalizing the classical braid's arrangement.
- $\mathcal{A} = \{H_1^0, \ldots, H_n^0\}, i = 1, \ldots, n$, is a generic arrangement in $\mathbb{C}^k$.
- $\mathbb{S}(\mathcal{A})$ denotes the spaces of parallel translates of $\mathcal{A}$.
- The closed subset of $\mathbb{S}(\mathcal{A})$ formed by the collection of hyperplanes which fail to form a generic arrangement is a union of hyperplanes D_L.
- Each hyperplane D_L corresponds to a subset $L = \{i_1, \ldots, i_{k+1}\} \subset [n]\ \{1, \ldots, n\}$ and it consists of n-tuples of translates of hyperplanes $H_1^0, \ldots, H_n^0$ in which translates of $H_{i_1}^0, \ldots, H_{i_{k+1}}^0$ fail to form a general position arrangement.
- The arrangement $\mathcal{B}(n, k, \mathcal{A})$ of hyperplanes D_L is called *discriminantal arrangement*.

Combinatorial Approach

- A permutation σ in a symmetric group S_n composed of disjoint transpositions is said to act strongly on the elements in the intersection lattice of $\mathcal{A}$ if it fixes non trivial collinearlities in $\mathcal{A}$.
- Six new lines $l'_1, l'_2, \ldots, l'_6$ added to the Pappus's configuration are obtained such that:
  - l'_i is the line $P\sigma.P$ where P is a multiplicity 2 intersection in the Pappus's configuration.
  - For any point P in intersection in the Pappus's configuration there exists exactly one line l'_i such that $P \in l'_i$.
- The arrangement formed by the new lines $l'_1, \ldots, l'_6$ is called σ completion of $\mathcal{P}^c_\infty$ (resp. $\mathcal{P}_\infty$) and denoted by $(\mathcal{P}^c_\infty)^\sigma$ (resp. $\mathcal{P}^\sigma_\infty$).

Intersection lattice of discriminantal arrangement

- An arrangement $\mathcal{A}$ is called a *very generic arrangement* if the number of intersections in the intersection lattice $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ is the largest possible between all the discriminantal arrangements $B(n,k,\mathcal{A}')$, when $\mathcal{A}'$ ranges between all generic arrangements of n hyperplanes in $\mathbb{R}^k(\mathbb{C}^k)$. Otherwise it is called a non very generic arrangement.
- An element X is called a simple intersection in $\mathcal{B}(n,k,\mathcal{A})$ if $X = \cap_{i=1}^{m} D_{L_i}, |L_i| = k+1$ and for every subset $I \subset [m], \mid I \mid \geq 2, \cap_{i \in I} D_{L_i} \neq D_K \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A})), K \subset [n], \mid K \mid > k+1$. In particular if $m > r$ we call X a non very generic simple intersection.

Intersection lattice of discriminantal arrangement

- The set containing all the permutations σ that acts strongly on $\mathcal{A}$ is denoted by $S_{\mathcal{A}}$.
- Since each collinearity condition in $\mathcal{A}$ corresponds to a simple intersection of rank 2 and multiplicity 3 of $\mathcal{B}(n, 3, \mathcal{A})$ then permutation σ acts strongly on $\mathcal{A}$ if and only if it fixes rank 2 and multiplicity 3 simple intersections of $\mathcal{B}(n, 3, \mathcal{A})$. We can say here that σ acts strongly on $\mathcal{B}(n, 3, \mathcal{A})$.

Intersection lattice of discriminantal arrangement

- If $\mathcal{P}_\infty$ and $\mathcal{P}_\infty^c$ satisfy the additional condition that the three collinearities of the classical Pappus's configuration are concurrent then for $\sigma \in S_{\mathcal{P}_\infty}$,
  1. $\mathcal{P}_\infty^c \cup \mathcal{P}_\infty^{c\,\sigma}$ is an arrangement with the minimum number of multiplicity 2 intersection if and only if $\sigma \in S_6$ acts strongly on $\mathcal{P}_\infty^c$,
  2. $\mathcal{P}_\infty \cup \mathcal{P}_\infty^\sigma$ is an arrangement with the maximum number of multiplicity 3 intersection otherwise.
- Two simple intersections of multiplicity 3 and rank 2 in $\mathcal{B}(n, 3, \mathcal{A})$ are called independent if they do not share any hyperplane.
- A simple intersection of multiplicity 3 in rank 2 is called purely dependent if it is intersection of 3 hyperplanes each one containing exactly one independent intersection.

Main Result

Let $\mathcal{B}(6,3,\mathcal{A})$ be a discriminantal arrangement with the maximum number of independent intersections in rank 2 $\sigma \in S_6$ acts strongly on $\mathcal{B}(6,3,\mathcal{A})$, then:

1. The arrangement $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the minimum number of intersections of multiplicity 2 if and only if there exists a purely dependent intersection fixed by σ in $\mathcal{B}(6,3,\mathcal{A})$ and $\mathcal{A}^\sigma$ is central.
2. $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma$ belongs to a simple intersection of multiplicity 4 in rank 3.

Conjecture

Let $\mathcal{B}(n, 3, \mathcal{A})$ be a discrimanantal arrangement with the maximum number of independent intersections in rank 2 and $\sigma \in S_n$ acts strongly on $\mathcal{B}(n, 3, \mathcal{A})$, then :

1. the arrangement $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the minimum number of intersections of multiplicity 2 if and only if purely dependent intersections in $(\mathcal{B}(n, 3, \mathcal{A}))$ are all fixed by σ and they are in maximum number.
2. $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma$ belongs to a is simple intersection X having the maximum multiplicity in rank $n-3$.

Thank You!!!