THE GENERALIZED SYLVESTER’S AND ORCHARD PROBLEMS VIA DISCRIMINANTAL ARRANGEMENT

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Objective

• We aim at elucidating the connection between the generalised Sylvester’s and Orchard Problem and the combinatorics of discriminantal arrangement $B(n, k, A)$.

• We answer the above question for a special case of arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$.

• An arrangement of lines is a finite collection of lines in a plane. The point where $r$ lines intersect is called a multiplicity of $r$ intersection.
The generalised Sylvester’s Problem

When posed in its dual form leads to the question: given an arrangement of \( n \) lines in \( \mathbb{C}^2 \) what is the minimum number of multiplicity 2 intersections.

**Figure 1:** Examples of arrangement with 5 lines
The generalised Orchard Problem

When posed in its dual form leads to the question: given an arrangement of \(n\) lines in \(\mathbb{C}^2\) what is the maximum number of multiplicity 3 intersections.

**Figure 2:** Examples of Orchard problem with \(n\) lines and multiplicity 3 intersections
Background

- Pappus’s configuration is an arrangement of 6 lines with 3-collinearity conditions.
- Pappus’s configuration with 3-collinearity conditions is denoted by $P_{\infty}$.
- Pappus’s configuration with 4-collinearity conditions is denoted by $P_{\infty}^c$.

Figure 3: Pappus’s configurations
In our problem we consider a Pappus’s configuration where the three classical collinearities are concurrent. Six new lines are added to the 6 lines in Pappus’s configuration in the following way to get the arrangement of 12 lines:

1. lines $l'_1$, $l'_2$, $l'_3$ are the three concurrent lines corresponding to the three Pappus’s collinearities;

2. lines $l'_4$, $l'_5$, $l'_6$ are added so that each one of them contains exactly two different multiplicity 2 intersection of $\mathcal{P}_\infty^C$ (resp. $\mathcal{P}_\infty$) and that each multiplicity 2 intersection is contained in only one line $l'_i$, $i = 1, \ldots, 6$. 
Arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$

Figure 4: Arrangement of 12 lines with 6 multiplicity 2 intersections in $\mathbb{P}^2\mathbb{R}$ where the black lines depict the Pappus’s configuration.
**Arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$**

**Figure 5:** Arrangement of 12 lines with 19 multiplicity 3 intersection in $\mathbb{P}^2\mathbb{R}$ where the black lines depict the Pappus's configuration.
The discriminantal arrangement \( B(n, k, \mathcal{A}) \) is an arrangement of hyperplanes, constructed from a generic arrangement \( \mathcal{A} \), generalizing the classical braid’s arrangement.

\( \mathcal{A} = \{H_1^0, \ldots, H_n^0\}, i = 1, \ldots, n \), is a generic arrangement in \( \mathbb{C}^k \).

\( \mathcal{S}(\mathcal{A}) \) denotes the spaces of parallel translates of \( \mathcal{A} \).

The closed subset of \( \mathcal{S}(\mathcal{A}) \) formed by the collection of hyperplanes which fail to form a generic arrangement is a union of hyperplanes \( D_L \).

Each hyperplane \( D_L \) corresponds to a subset \( L = \{i_1, \ldots, i_{k+1}\} \subset [n] \{1, \ldots, n\} \) and it consists of \( n \)-tuples of translates of hyperplanes \( H_1^0, \ldots, H_n^0 \) in which translates of \( H_{i_1}^0, \ldots, H_{i_{k+1}}^0 \) fail to form a general position arrangement.

The arrangement \( B(n, k, \mathcal{A}) \) of hyperplanes \( D_L \) is called discriminantal arrangement.
Combinatorial Approach

- A permutation $\sigma$ in a symmetric group $S_n$ composed of disjoint transpositions is said to act strongly on the elements in the intersection lattice of $\mathcal{A}$ if it fixes non trivial collinearities in $\mathcal{A}$.
- Six new lines $l'_1, l'_2, \ldots, l'_6$ added to the Pappus’s configuration are obtained such that:
  - $l'_i$ is the line $P_{\sigma}.P$ where $P$ is a multiplicity 2 intersection in the Pappus’s configuration.
  - For any point $P$ in intersection in the Pappus’s configuration there exists exactly one line $l'_i$ such that $P \in l'_i$.
- The arrangement formed by the new lines $l'_1, \ldots, l'_6$ is called $\sigma$ completion of $\mathcal{P}_\infty^c$ (resp. $\mathcal{P}_\infty$ ) and denoted by $(\mathcal{P}_\infty^c)^\sigma$ (resp. $\mathcal{P}_\infty^\sigma$ ).
• An arrangement $\mathcal{A}$ is called a *very generic arrangement* if the number of intersections in the intersection lattice $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}))$ is the largest possible between all the discriminantal arrangements $\mathcal{B}(n, k, \mathcal{A}')$, when $\mathcal{A}'$ ranges between all generic arrangements of $n$ hyperplanes in $\mathbb{R}^k(\mathbb{C}^k)$. Otherwise it is called a non very generic arrangement.

• An element $X$ is called a simple intersection in $\mathcal{B}(n, k, \mathcal{A})$ if $X = \bigcap_{i=1}^{m} D_{L_i}, |L_i| = k + 1$ and for every subset $I \subset [m], |I| \geq 2, \bigcap_{i \in I} D_{L_i} \neq D_K \in \mathcal{L}(\mathcal{B}(n, k, \mathcal{A})), K \subset [n], |K| > k + 1$. In particular if $m > r$ we call $X$ a non very generic simple intersection.
The set containing all the permutations $\sigma$ that acts strongly on $\mathcal{A}$ is denoted by $S_{\mathcal{A}}$.

Since each collinearity condition in $\mathcal{A}$ corresponds to a simple intersection of rank 2 and multiplicity 3 of $\mathcal{B}(n, 3, \mathcal{A})$ then permutation $\sigma$ acts strongly on $\mathcal{A}$ if and only if it fixes rank 2 and multiplicity 3 simple intersections of $\mathcal{B}(n, 3, \mathcal{A})$. We can say here that $\sigma$ acts strongly on $\mathcal{B}(n, 3, \mathcal{A})$. 
Intersection lattice of discriminantal arrangement

• If $P_\infty$ and $P_\infty^c$ satisfy the additional condition that the three collinearities of the classical Pappus’s configuration are concurrent then for $\sigma \in S_{P_\infty}$,

1. $P_\infty^c \cup P_\infty^c \sigma$ is an arrangement with the minimum number of multiplicity 2 intersection if and only if $\sigma \in S_6$ acts strongly on $P_\infty^c$,

2. $P_\infty \cup P_\infty^\sigma$ is an arrangement with the maximum number of multiplicity 3 intersection otherwise.

• Two simple intersections of multiplicity 3 and rank 2 in $B(n, 3, A)$ are called independent if they do not share any hyperplane.

• A simple intersection of multiplicity 3 in rank 2 is called purely dependent if it is intersection of 3 hyperplanes each one containing exactly one independent intersection.
Main Result

Let $\mathcal{B}(6, 3, \mathcal{A})$ be a discriminantal arrangement with the maximum number of independent intersections in rank 2 $\sigma \in S_6$ acts strongly on $\mathcal{B}(6, 3, \mathcal{A})$, then:

1. The arrangement $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the minimum number of intersections of multiplicity 2 if and only if there exists a purely dependent intersection fixed by $\sigma$ in $\mathcal{B}(6, 3, \mathcal{A})$ and $\mathcal{A}^\sigma$ is central.

2. $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma$ belongs to a simple intersection of multiplicity 4 in rank 3.
Conjecture

Let $\mathcal{B}(n, 3, \mathcal{A})$ be a discriminantal arrangement with the maximum number of independent intersections in rank 2 and $\sigma \in S_n$ acts strongly on $\mathcal{B}(n, 3, \mathcal{A})$, then:

1. the arrangement $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the minimum number of intersections of multiplicity 2 if and only if purely dependent intersections in $(\mathcal{B}(n, 3, \mathcal{A}))$ are all fixed by $\sigma$ and they are in maximum number.

2. $\mathcal{A} \cup \mathcal{A}^\sigma$ is an arrangement with the maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma$ belongs to a simple intersection $X$ having the maximum multiplicity in rank $n - 3$. 

Thank You!!!