Peeking at quantum gravity with self-overlapping curves

Women at the Intersection of Mathematics & Theoretical Physics

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OIST

What is Quantum Gravity?

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Within quantum field theory, we would like to write and solve:

$$Z = \int_{\mathcal{M}} \mathcal{D}g_{\mu\nu} \mathcal{D}\Phi \exp(-S[g_{\mu\nu}, \Phi])$$

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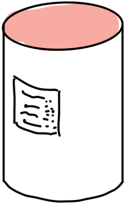
What are $\mathcal{D}$, $S[\cdot]$, Z?

Postulate (['t Hooft 1993, Susskind 1995]):

Quantum Gravity ($D+1$ dimensions)

$=$ **Quantum Theory (D dimensions)**



One approach: Holography

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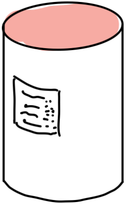

Hints for the emergence of gravity:

- Black hole entropy [Bekenstein 1972]:

$$S_{BH} = \frac{c^3}{4G\hbar} A_{\text{horizon}}$$

- Laws of black hole thermodynamics [Bardeen, Bekenstein, Carter, Hawking 1973]

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A simple model of holography

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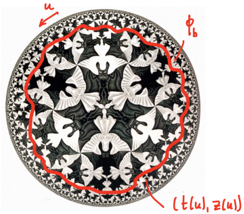
$$Z = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \exp\left(\frac{1}{16\pi G_N}\int \mathrm{d}^2x\sqrt{g}\phi(R+2) + \frac{\phi_b}{8\pi G_N}\oint \mathrm{d}sk\right) .$$



Source: math.slu.edu

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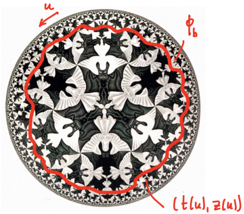

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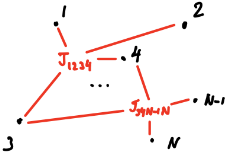

Boundary: **Sachdev-Ye-Kitaev model** ($D = 1$)

$$H = \sum_{1\leq i<j<k<l\leq N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l, \quad \left\langle J_{ijkl}^2\right\rangle \propto \frac{J^2}{N^3} .$$

Same effective action on the **boundary** (review [Mertens 2022]):

$$\frac{\phi_b}{8\pi G_N} \oint_{S^1} \mathrm{d}u Sch[t,u] \, ,$$

with $t : S^1 \to S^1,\ t'(u) > 0$ (**reparametrization**) and

$$Sch[t,u] = \left(\frac{t''(u)}{t'(u)}\right)' - \frac{1}{2}\left(\frac{t''(u)}{t'(u)}\right)^2 .$$

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But no discrete energy spectrum...

Immersions of the disk

We are interested in metrics on the disk.

Conformal gauge:

$$\mathrm{d}s^2 = e^{2\Sigma}|\mathrm{d}z|^2 \ , \qquad z = x + iy \ .$$

Metrics of constant curvature:

$$4\partial_z\partial_{\bar{z}}\Sigma = -\kappa e^{2\Sigma} \ , \quad \kappa = \pm 1, 0 \ .$$

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Theorem

a) *Let $\Sigma_b : S^1 \to \mathbb{R}$ be a continuous function defined on the boundary of the disk. Then there exists a unique solution $\Sigma \in C^\infty(D)$ of the Liouville equation such that $\Sigma = \Sigma_b$ on the boundary.*

b) *Assuming F holomorphic, the most generic solution (up to disk automorphisms, $PSL(2,\mathbb{R})$) takes the form:*

$$e^{\Sigma(z)} = \frac{2|F'(z)|}{1 + \kappa|F(z)|^2}\,.$$

We parametrize metrics on the disk $\mathcal{D}$:

$$\mathrm{d}s^2 = \frac{4|F'(z)|^2}{\left(1+\kappa|F(z)|^2\right)^2}|\mathrm{d}z|^2, \quad \begin{cases} F:\mathcal{D}\to H^2,\mathbb{R}^2,S^2 \text{ holomorphic,} \\ F'(z)\neq 0\ \forall z\in\mathcal{D}. \end{cases}$$

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If F is globally injective ($\implies F'(z) \neq 0$): **embedding**.
If F is locally injective ($\iff F'(z) \neq 0$): **immersion**.

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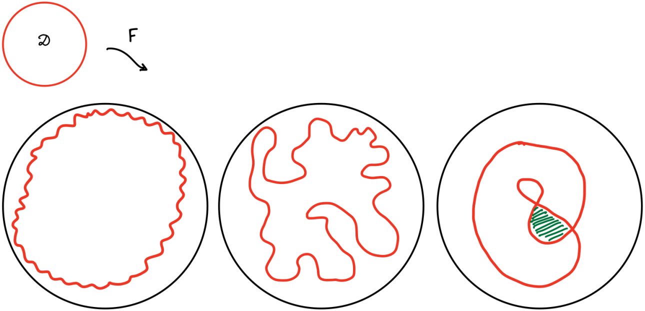

Reparametrization embedding – General embedding (self-avoiding) – Immersion

Questions

- How does considering immersions change the previous results?
- What are the properties of those immersions?
  Number of self-overlaps, fractals,...?
- Of their boundaries?
  Do they characterise the whole immersion?
  Minimal combinatorial properties that lead to an immersion?

Self-overlapping curves = curves that are boundary of an immersed disk.

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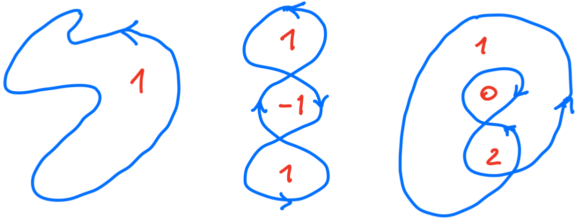
Graver & Cargo [2011] solved the problem with graph theory (covering graph).

- Turning number (index)

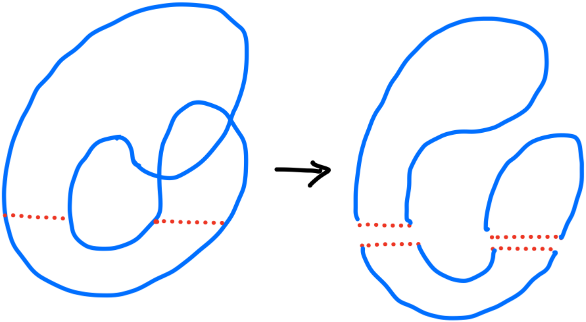
$$\text{turn}(\gamma) = \frac{1}{2\pi}\oint_\gamma k\,\mathrm{d}s = 1\,, \qquad k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}\,.$$

- Winding number (number of overlaps)

$$\text{wind}_\gamma(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{\gamma'(t)}{\gamma(t) - z_0}\mathrm{d}t \geq 0\,.$$

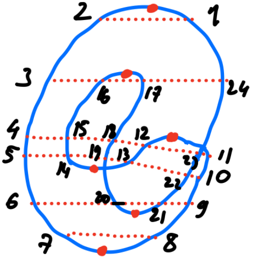


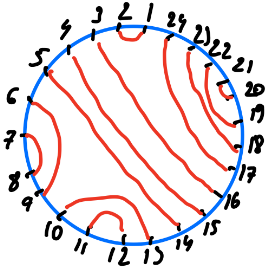
Curves that can be decomposed into simple curves through well-chosen cuts.



Self-overlapping curves: Maximally Planar Matchings

[Bonsma, Breuer 2012] Mapping the curve, together with good Blank cuts*, to a chordal graph, the problem of counting inequivalent disks corresponds to counting **Maximum Independent Sets** in the circle graph (for n vertices of the circle graph, $\mathcal{O}(n^2)$).





*such that the cut with the slices of the curve form a simple curve

Self-overlapping curves: Inequivalent disks

Examples of boundary curves that don't have a unique holomorphic extension:

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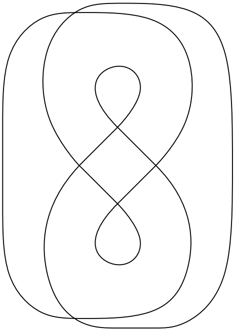
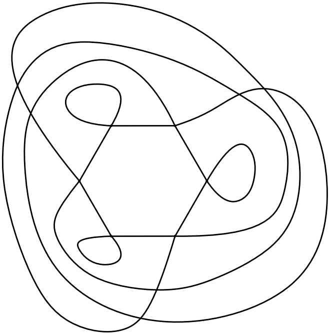


(a) Milnor (2 disks)



(b) Bennequin (5 disks)

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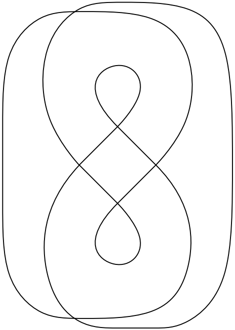

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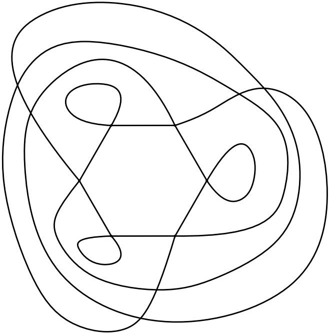
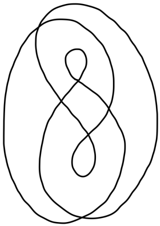


(b) Bennequin (5 disks)

NB: They can also be glued together.

Milnor's doodle



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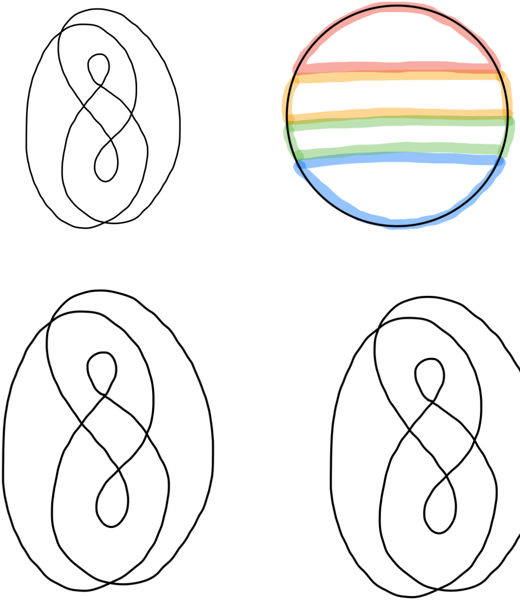

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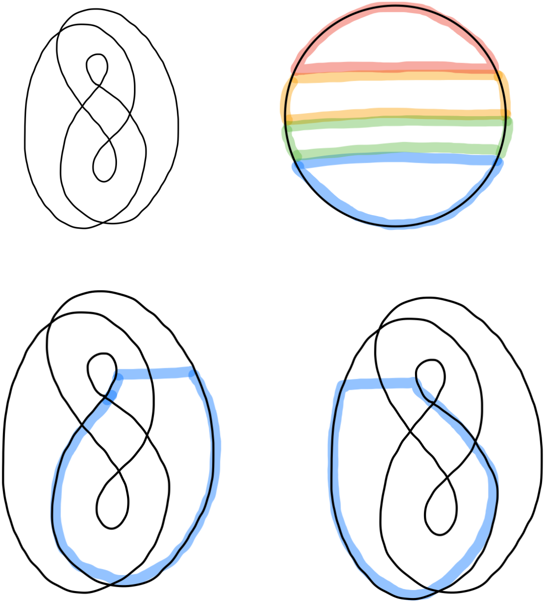

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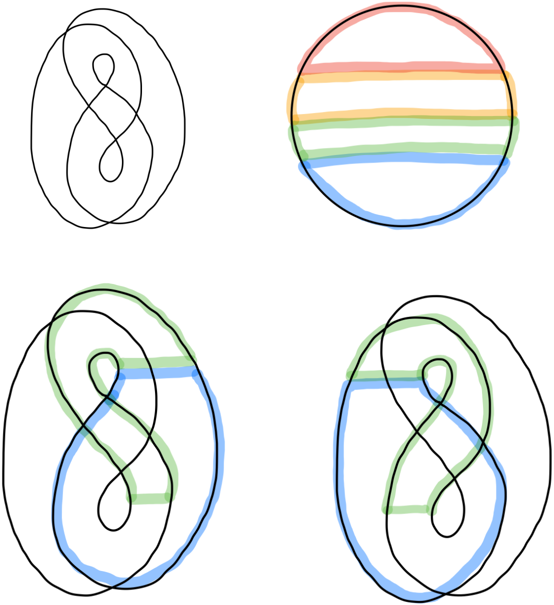

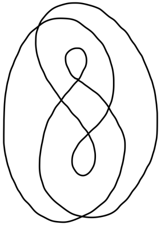

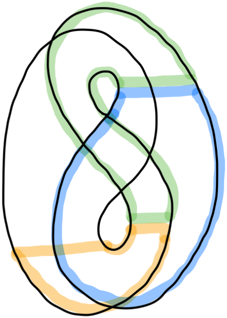

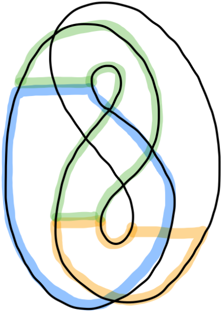

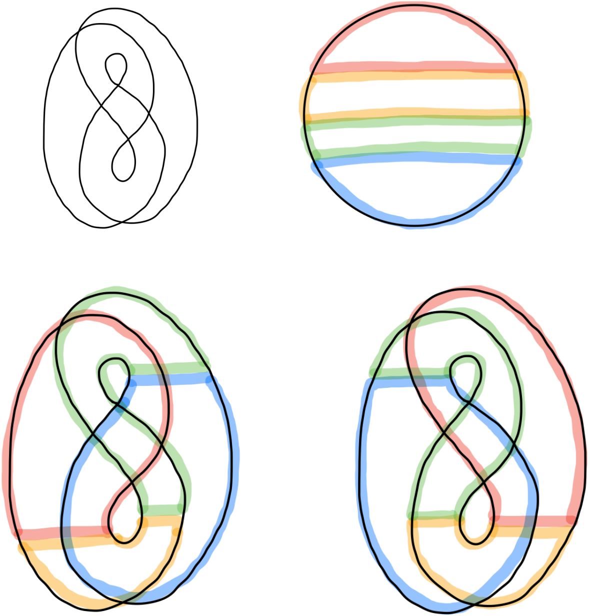

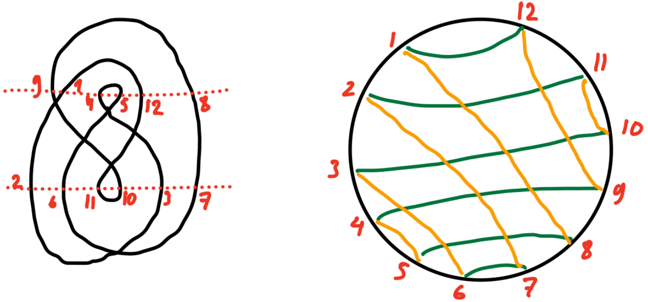

Figure: Minimal number of cuts and the associated "good" pairings.

Monte Carlo: 2D quantum "flat" gravity

Random samples of immersed disks in $\mathbb{R}^2$ (i.e. random flat metrics on the disk)

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1) Generate random Gaussian field $\Sigma_D(\theta)$: 2 parameters: $\{N, \sigma\}$
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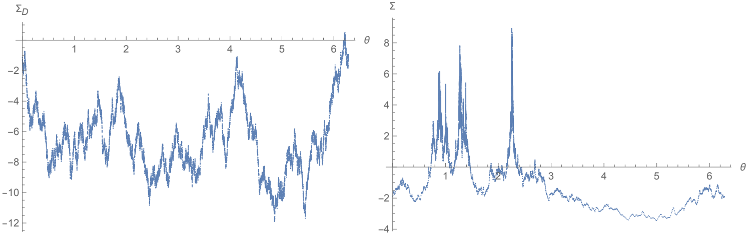
2) $\ell = \int \mathrm{d}\theta e^{\Sigma_D(\theta)}$
   (arclength-uniform: $\mathrm{d}\vartheta = \frac{2\pi}{\ell} e^{\Sigma_D(\theta)} \mathrm{d}\theta$)
3) Redefine $\Sigma(\vartheta) = -\Sigma_D(\theta(\vartheta)) + 2\log(\ell/2\pi)$



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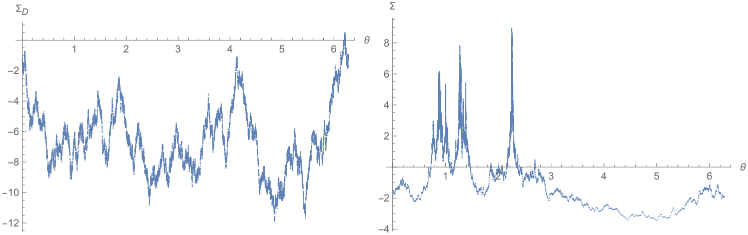

Σ has an action invariant under $PSL(2, \mathbb{R})$.

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Random samples of immersed disks in $\mathbb{R}^2$ (i.e. random flat metrics on the disk)

4) Analytic continuation:

$$H(z)|_{z=e^{i\vartheta}} = \Sigma(\vartheta) + i\Gamma(\vartheta) \,, \quad \Gamma(\vartheta) = \frac{1}{2\pi} \mathrm{P} \int \mathrm{d}\vartheta' \, \frac{\Sigma(\vartheta')}{\tan \frac{\vartheta' - \vartheta}{2}} \,,$$

5) Integrate the exponential of its analytic continuation:

$$F(z)|_{z=e^{i\vartheta}} = i \int_0^{\vartheta} \mathrm{d}\vartheta' e^{i\vartheta'} \exp\Big[H(e^{i\vartheta'}) \Big] \,, \quad \text{(gauge: } F(1) = 0) \,.$$

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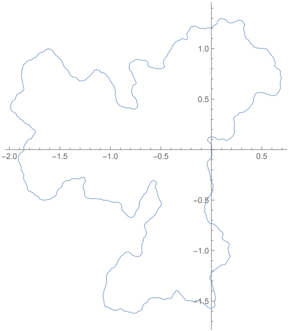
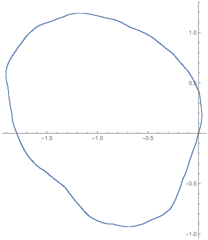
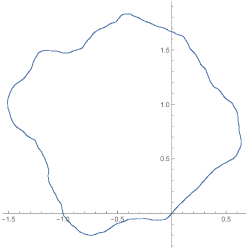


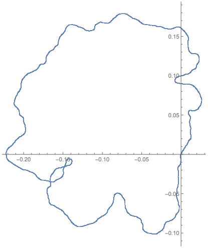
Monte Carlo: Samples



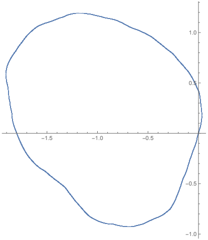
(a) $\sigma = 0.5$



(b) $\sigma = 1.5$



(c) $\sigma = 3$



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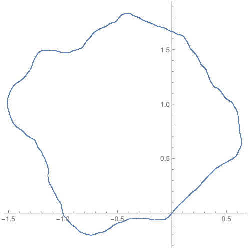

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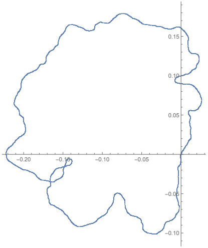

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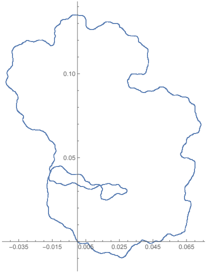
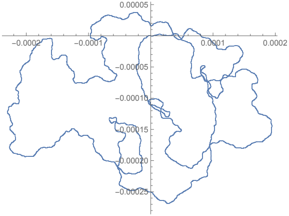
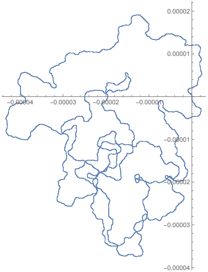


(d) $\sigma = 4$



(e) $\sigma = 7$



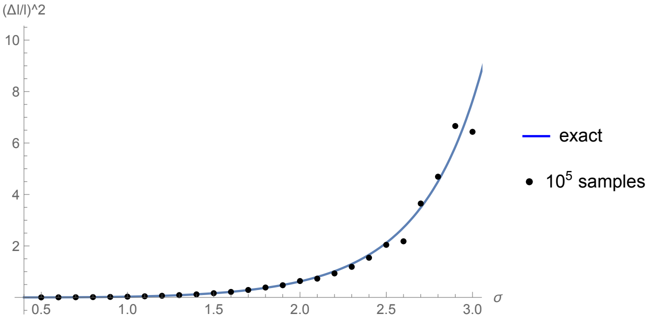
(f) $\sigma = 8$

$$\ell = \int \mathrm{d}\theta e^{\Sigma(\theta)} \;, \quad \langle \ell \rangle = 2\pi \;,$$

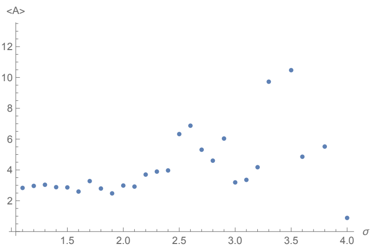
$$\left\langle \ell^2 \right\rangle = \frac{4\pi^2}{\sigma} \exp\left(-\frac{\pi\sigma^2}{12}\right) \mathrm{Erfi}\left(\frac{\sqrt{\pi}\sigma}{2}\right) \;,$$

$$\Delta\ell = \sqrt{\langle \ell^2 \rangle - \langle \ell \rangle^2} \;,$$

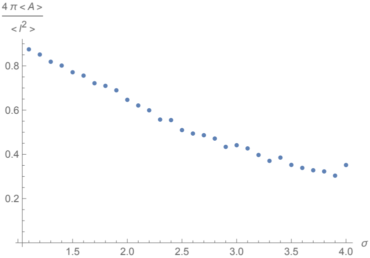
$$\mathrm{Erfi}(z) = \frac{-2i}{\sqrt{\pi}} \int_0^z \mathrm{d}t \; e^{t^2} \;.$$



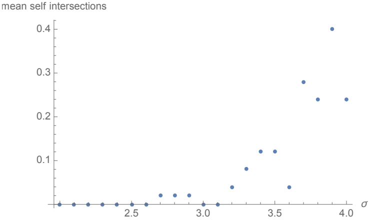
Monte Carlo: Areas, self-intersections and overlaps (100 samples)



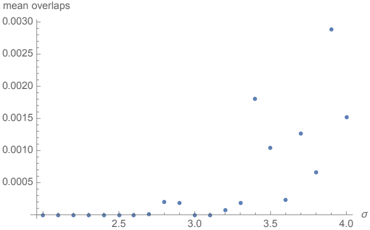
(a) $\langle A \rangle$



(b) $4\pi \langle A \rangle \,/\, \langle l^2 \rangle$



(c) Self-intersections.



(d) Self-overlaps.

Random metrics (with fixed curvature) on the disk
correspond to self-overlapping curves.

Conclusions and Perspectives

Random metrics (with fixed curvature) on the disk
correspond to self-overlapping curves.

- What is this kind of new random object?
  (fixed length, extrinsic curvature, order parameter...)
- Implement counting and identification of distinct immersions.
  (faster algorithms using minimal number of cuts?)
- Partition function.
- Hyperbolic case, other topologies...

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Thank you!

Self-overlapping curves: technical results

Theorem (Graver, Cargo 2011)

An oriented normal curve γ, with $0 \leq wind_\gamma(f) \leq 2$, admits a unique extension if:

(i) *the number of faces with $wind_\gamma(f) = 2$ equals the number of faces with $wind_\gamma(f) = 0$,*

(ii) *all faces with $wind_\gamma(f) = 2$ have boundaries of even length.*

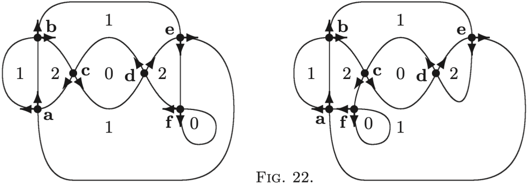

FIG. 22.

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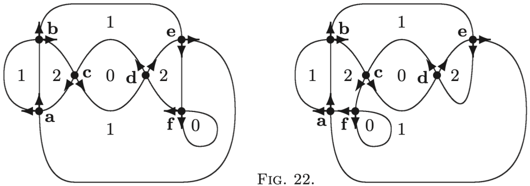

FIG. 22.

Theorem (Shor, Van Wyk 1992)

The number of incompatible decompositions is equal to the number of combinatorially inequivalent **constrained Delaunay triangulations**.