Irreducible Specht modules for symmetric groups and beyond

Louise Sutton

Women at the Intersection of Mathematics and Theoretical Physics Meet in Okinawa

OIST

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subject to the relations

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FIELDS OF CHARACTERISTIC ZERO: e.g. $\mathbb{Q}$, $\mathbb{C}$.

FIELDS OF PRIME CHARACTERISTIC: e.g. finite fields $\mathbb{Z}/p\mathbb{Z}$, p a prime.

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Open Problem

Can we explicitly describe the irreducible modules of $\mathfrak{S}_n$?
What are their dimensions? What are their bases?

Introducing combinatorics

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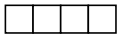
(4) 

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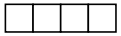
(4)		
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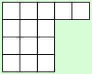
Let $p = 3$. $[(5,3^3)] =$  is neither 3-regular nor 3-restricted

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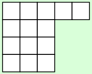
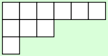
Let $p = 3$. $[(5, 3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6, 3, 1)] =$  is 3-regular but not 3-restricted.

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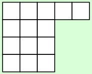
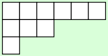
Let $p = 3$. $[(5,3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6,3,1)] =$  is 3-regular but not 3-restricted.

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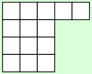
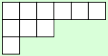
Let $p = 3$. $[(5, 3^3)] =$  is neither 3-regular nor 3-restricted,

but $[(6, 3, 1)] =$  is 3-regular but not 3-restricted.

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Open Problem

What are the dimensions and bases of irreducible $\mathfrak{S}_n$*-modules?*

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A LONG-STANDING CONJECTURE

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Aim

First step: *classify the irreducible Specht modules in positive characteristic.*

Hook lengths

Let λ be a partition, and $[\lambda]$ be its Young diagram.

The **hook length** of a box $(a, b) \in [\lambda]$ is

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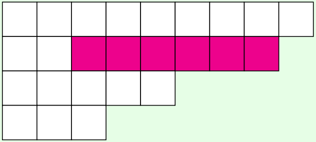
$h_{ab}^{\lambda} =$  $= 6$

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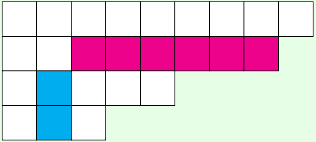

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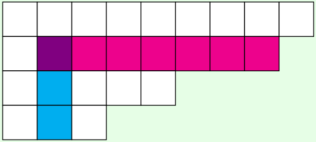
$h_{ab}^{\lambda} =$  $= 6 + 2 + 1 = 9.$

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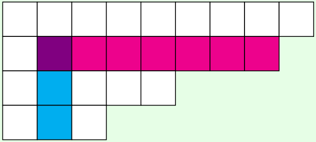

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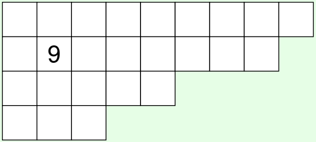

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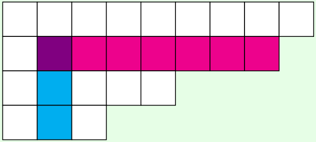

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								1
	9						1	
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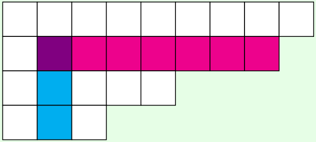

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	9					2	1	
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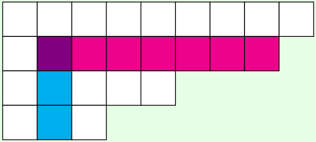
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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							3	1
	9				3	2	1	
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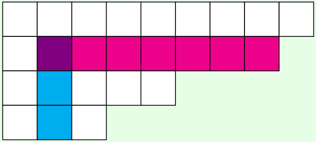
$h^{\lambda}_{ab} =$  $= 6 + 2 + 1 = 9.$

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						4	3	1
	9				3	2	1	
		4	2	1				
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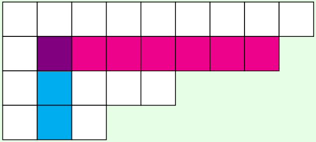
$h_{ab}^{\lambda} =$  $= 6 + 2 + 1 = 9.$

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	9			5	3	2	1	
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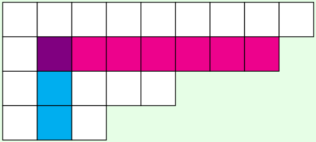

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12	11	10	8	7	5	4	3	1
10	9	8	6	5	3	2	1	
6	5	4	2	1				
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17	14	10	8	7	5	4	3	2	1
11	8	4	2	1					
8	5	1							
6	3								
5	2								
4	1								
2									
1									

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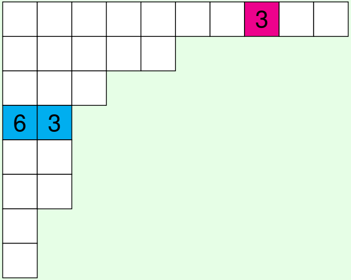

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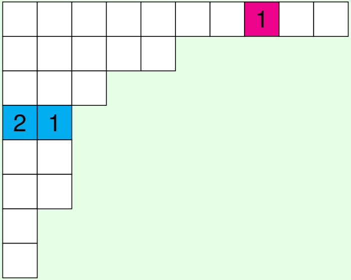

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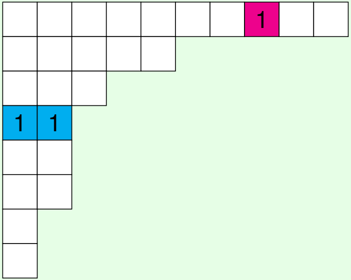

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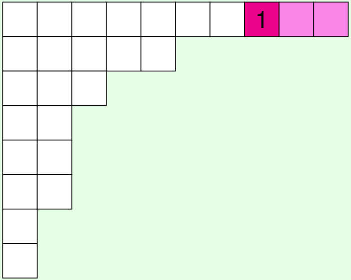

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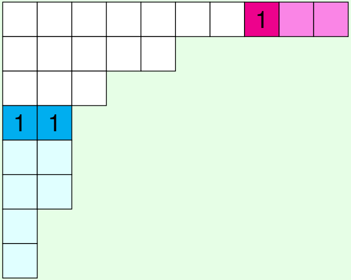

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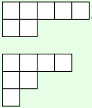
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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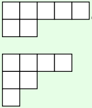
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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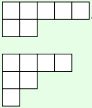
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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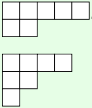
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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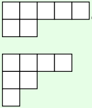
$((5,2),(4,2,1))$ is a bipartition of 14, with Young diagram .

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Can we classify irreducible Specht modules $S(\lambda, \mu)$?

Conjectural classification

Conjecture (Joint work with Matthew Fayers)

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