Partial permutohedra

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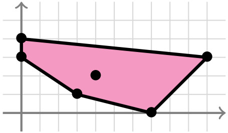
Polytopes

The **convex hull** of a set $S \subseteq \mathbb{R}^d$ is the smallest convex set containing S. We denote it by $\mathrm{Conv}(S)$.

Example. The convex hull of $\{(3,1),(4,2),(7,0),(10,3),(0,3),(0,4)\}$ is:



A **polytope** is the convex hull of a finite set.

An invitation to measure polytopes

The geometry of linear optimization is understood by studying polytopes.

The number of k-dimensional regions of a hyperplane arrangement in $\mathbb{R}^n$ is equal to the number of $(n-k)$-dimensional faces of a zonotope.

Using the rich correspondence between geometry of toric varieties and combinatorics of convex polytopes, Batyrev constructed classes Calabi-Yau manifolds as hypersurfaces of toric varieties and proved the mirror symmetry conjecture for smooth toric varieties.

The n-dimensional associahedron is a polytope whose number of vertices equals the $(n+1)$-th Catalan number. Recently, Mizera and Arkani-Hamed–Bai–He–Yan showed that this polytope plays a central role in the theory of scattering amplitudes.

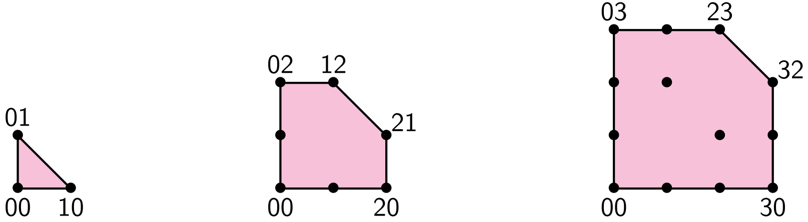
Given a polytope, how many vertices does it have? How many faces? What's its volume?

Theorem. (Dryer 83, Linial 86, Dryer–Frieze 88) Computing any of these values is $\#P$-hard.

Partial permutohedra

Definition. (Heuer–Striker '22) Given $d, n \in \mathbb{Z}_{>0}$, the **partial permutohedron** $\mathcal{P}(d, n)$ is the convex hull of all vectors in $\{0, 1, \ldots, n\}^d$ whose nonzero entries are distinct.

Example. Let $d = 2$.



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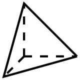
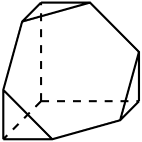
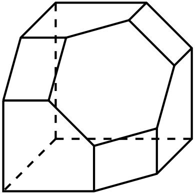
Example. Let $d = 3$.







Number of faces

The **f-polynomial** of a d-dimensional polytope $\mathcal{P}$ is $f_{\mathcal{P}}(t) = \sum_{i=0}^{d} f_i(\mathcal{P})\, t^i$, where $f_i(\mathcal{P})$ denotes the number of i-dimensional faces of $\mathcal{P}$.

The **Eulerian polynomial** is $A_d(t) = \sum_{i=0}^{d-1} A(d,i)\, t^i$, where $A(d,i)$ is the number of permutations in S_d with exactly i descents.

A **descent** of w is a position $i < n$ with $w(i) > w(i+1)$. For example, $w = 3452167$ has descents at positions 3 and 4.

Theorem. (BCCDEHI '22+) The f-polynomial of $\mathcal{P}(d,n)$ with $n \leq d$ is

$$f_{\mathcal{P}(d,n)}(t) = 1 + \sum_{i=0}^{n-1} \sum_{j=1}^{d-i} \binom{d}{i} A_i(t+1)\,(t+1)^j.$$

Volume of $\mathcal{P}(d, n)$ with $n \geq d - 1$

To simplify volume expressions, we use the normalized volume defined such that $\mathrm{Vol}([0,1]^m) = m!$.

Denote by $v(d, n)$ the normalized volume of $\mathcal{P}(d, n)$.

Theorem. (BCCDEHI '22+) For any d and n with $n \geq d - 1$, the normalized volume of $\mathcal{P}(d, n)$ is given recursively by

$$v(d, n) = (d-1)! \sum_{k=1}^{d} k^{k-2} \frac{v(d-k, n-k)}{(d-k)!} \left(kn - \binom{k}{2} \right) \binom{d}{k},$$

with the initial condition $v(0, n) = 1$.

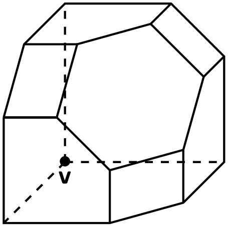
Examples.

$$\begin{aligned}
v(1,n) &= & & & & n, \\
v(2,n) &= -1 & & & & +2n^2, \\
v(3,n) &= -6 & -9n & & & +6n^3, \\
v(4,n) &= -54 & -96n & -72n^2 & & +24n^4, \\
v(5,n) &= -840 & -1350n & -1200n^2 & -600n^3 & +120n^5.
\end{aligned}$$

Ideas behind the proof

Lemma. Let $\mathbf{v}$ be a vertex of $\mathcal{P}(d,n)$. For each facet $\mathcal{F}$ that does not contain $\mathbf{v}$, form the pyramid $\mathrm{Pyr}(\mathcal{F},\mathbf{v})$. The collection of these pyramids for all such facets gives a polyhedral subdivision of $\mathcal{P}(d,n)$, and thus

$$\mathrm{Vol}(\mathcal{P}(d,n)) = \sum_{\substack{\text{facets } \mathcal{F} \\ \mathbf{v}\notin\mathcal{F}}} \mathrm{Vol}\left(\mathrm{Pyr}(\mathcal{F},\mathbf{v})\right).$$



Let
$\Pi_k = \mathrm{Conv}\{(w(1),\ldots,w(k)) \mid w \in S_k\}$.

Each facet not containing $\mathbf{0}$ is congruent to either a Cartesian product $\Pi_k \times \mathcal{P}(d-k, n-k)$ or Π_d.

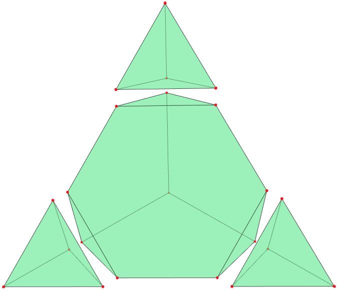
This method was used to obtain the volume of $\mathcal{P}(d,n)$ for the case $n = d-1$ by Amanbayeva–Wang and Strong.

Next, we fix n and vary d.

Volume of $\mathcal{P}(d, 2)$

The following was conjectured in (Heuer–Striker '21).

Theorem. (BCCDEHI '22+) For any d, the normalized volume of $\mathcal{P}(d, 2)$ is $v(d, 2) = 3^d - d$.

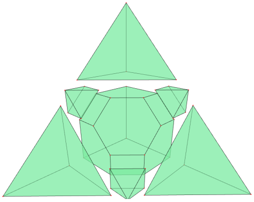


$\mathcal{P}(d, 2)$ is equal to $3\,\mathrm{Conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\}$ with d simplices removed.

Volume of $\mathcal{P}(d,3)$

Theorem. (BCCDEHI '22+) For any d, the normalized volume of $\mathcal{P}(d,3)$ is

$$v(d,3) = 6^d - d\,3^d - (d-1)\binom{d}{2}.$$



Theorem. (BCCDEHI '22+) For any d, the normalized volume of $\mathcal{P}(d,4)$ is

$$v(d,4) = 10^d - d\,6^d - \frac{d(d-1)(d-3)}{6}\,3^d - (3d^2 - 6d + 1)\binom{d}{3}.$$

Thank you!