Diagrammatic left canonical form of braids & applications

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joint w/ Michele Capovilla-Searle & Rebecca Sorsen

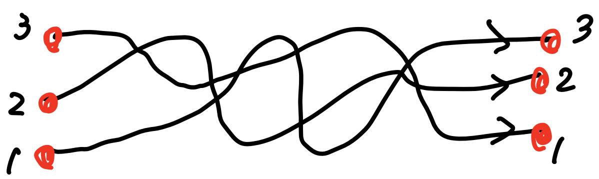
Braid gp $B_n = \left\{ \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2 \end{array} \right\}$

Q (Word problem)
Given β & $\beta' \in B_n$ determine $\beta = \beta'$ or $\beta \neq \beta'$.

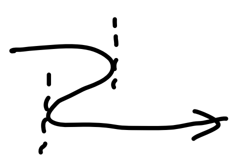
Q (Conjugacy problem)
—————— " —————— $\beta \underset{conj}{\sim} \beta'$ or $\beta \not\sim \beta'$.

Def: $\exists \gamma \in B_n$ s.t. $\beta' = \gamma \beta \gamma^{-1}$

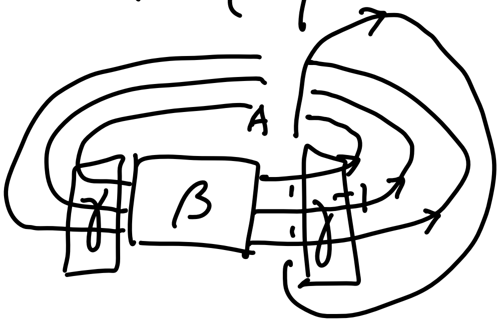
Geometrically

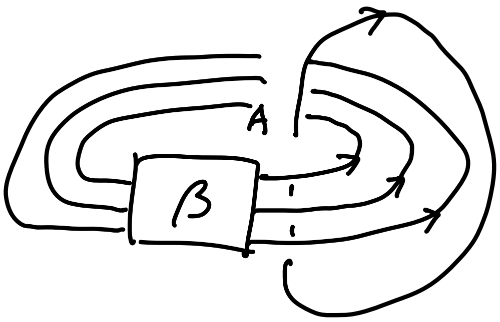


don't allow



- $\beta = \beta'$ (isotopic when end pts are fixed)
- $\beta \sim \beta'$ ($\hat{\beta} \cup A = \hat{\beta}' \cup A$) as oriented links





Birman-Ko-Lee

Given $\beta \in B_n$ there is a special factorization $LCF(\beta)$ of β s.t.

$$LCF(\beta) = LCF(\beta') \iff \beta = \beta'$$

exactly the same — up to braid relations

Today, study $LCF(\beta)$ via diagrams.

originaly very algebraic.

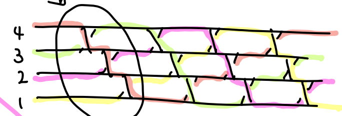
$$LCF(\beta) = \delta^r A_1 \cdots A_k$$

a factorization of β

$$\delta, A_1, \cdots, A_k \in CnFct(B_n)$$

Def δ = fundamental element $\in B_n$

$= \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1$



$\delta^n = \Delta^2$ full twist

δA_1, $A_i A_{i+1}$ are all maximally left weighted.

$r =: \inf(\beta)$ $\quad r+k =: \sup(\beta)$

$$\mathrm{ArtGen}(B_n) \subset \mathrm{BdGen}(B_n) \subset \mathrm{CnFct}(B_n) \subset B_n^+ \subset B_n$$

$\left\{ \begin{array}{c} \text{Artin Gens} \\ \sigma_1, \cdots, \sigma_{n-1} \end{array} \right\}$

$n-1$

$\left\{ \begin{array}{c} \text{Band Gens} \\ a_{ij} \\ 1 \le i < j \le n \end{array} \right\}$

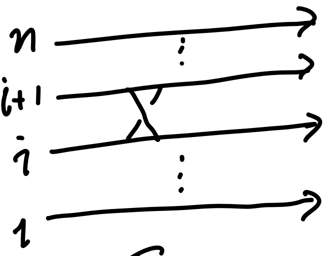
$\binom{n}{2} = \frac{n(n-1)}{2}$

Catalan #

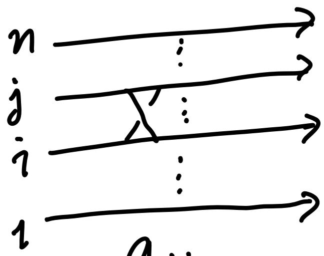
$C_n = \frac{(2n)!}{n!\,(n+1)!}$

B_n^+: monoid of bd gens

∞ ∞



σ_i



a_{ij}

$\underline{\text{BKL}}$ $\mathrm{CnFct}(B_n) \underset{df}{=} \left\{ \beta \in B_n^+ \;\middle|\; \begin{array}{l} \exists P, Q \in B_n^+ \text{ s.t.} \\ P \beta Q = \delta \end{array} \right\}$

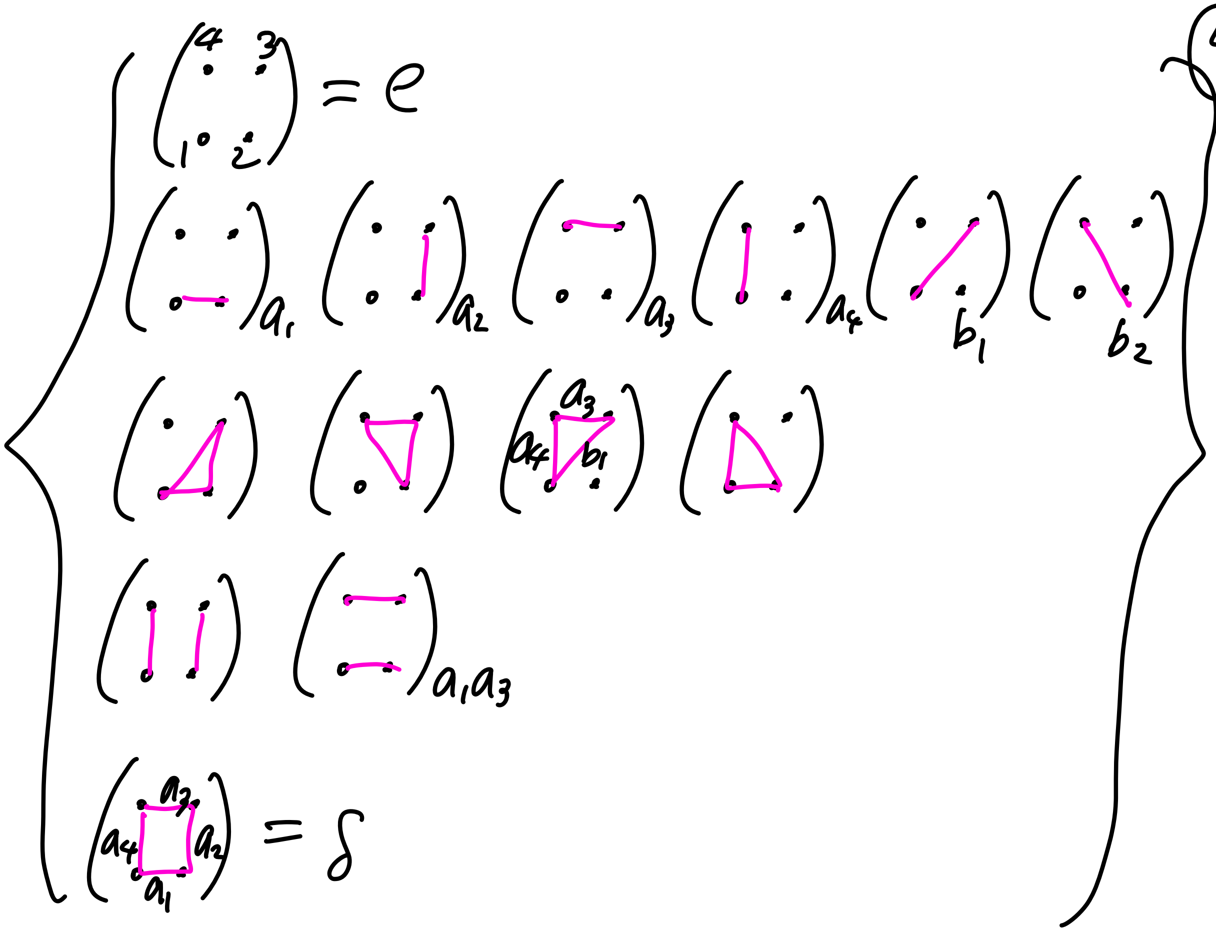
Implicit in BKL

$\mathrm{CnFct}(B_n) \underset{1:1}{\longleftrightarrow} \{\text{crossingless diagrams for } D_n\}$

Ex

$C_4 = 14$ ✓

$\mathrm{CnFct}(B_4) \longleftrightarrow$



Def Partial order "$\prec$" on $\mathrm{CnFct}(B_n)$.

$A, B \in \mathrm{CnFct}(B_n)$

$A \prec B \overset{\text{df}}{\iff} \exists Q \in \mathrm{CnFct}(B_n)$ s.t $AQ = B$

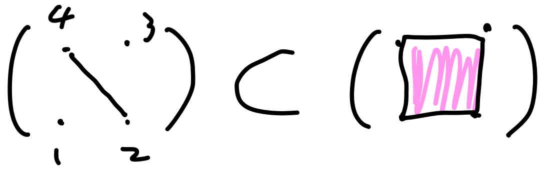
algebraic
hard to detect

abstruct.

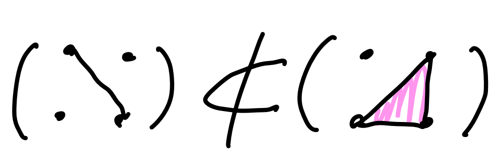
Th (CKS)

$A \prec B \iff cv(A) \subset cv(B)$.

visual

Ex 

$a_{24} \prec \sigma_3\sigma_2\sigma_1 = \delta$



$a_{24} \not\prec \sigma_2\sigma_1$

Def relation ($\Rightarrow$) btw words.

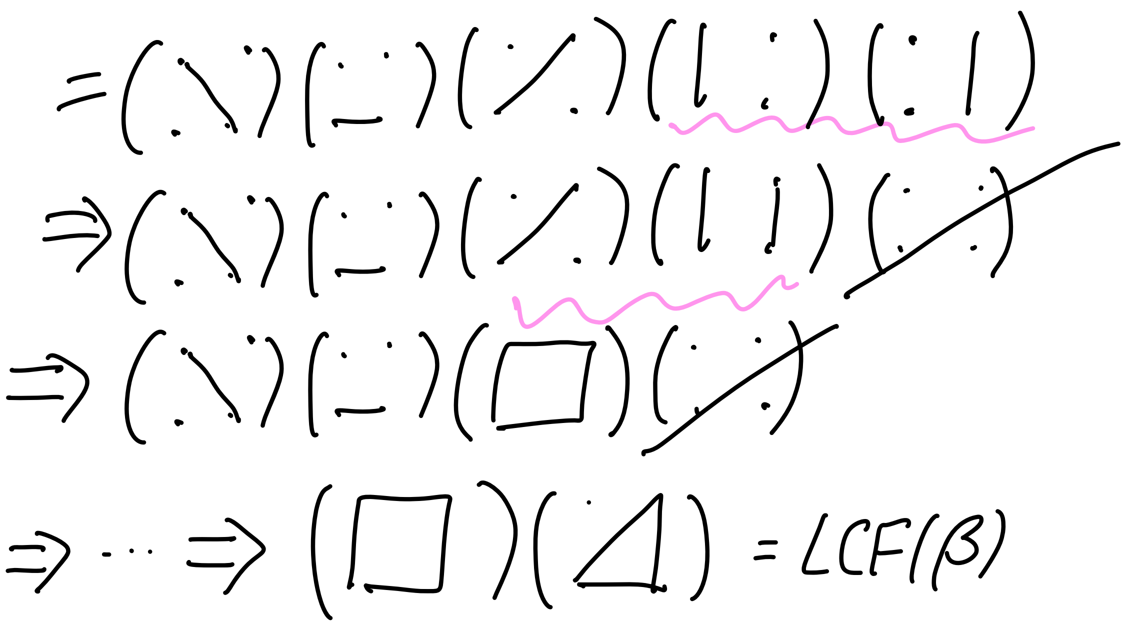
$A, B, A', B' \in CF(B_n)$

$AB \Rightarrow A'B' \overset{}{\underset{df}{\Longleftrightarrow}} AB = A'B'$ as braids

$A \prec A'$

($A'B'$ is more left weighted than AB.)

Ex $\beta = (\sigma_3\sigma_2\overline{\sigma_3})\ \sigma_1\ (\sigma_2\sigma_1\sigma_2^{-1})(\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1})\sigma_2$



$= LCF(\beta)$

Applications

Alg geom

$\{\hat{\beta} \mid \beta \in QP\} = \{S^3 \cap \text{cplx curve in } \mathbb{C}^2\}$

normally generated monoid of $w\sigma_i w^{-1}$

〃 a_{ij}

〃 σ_i

4D

$$\begin{array}{ccccccc} & & QP & \subset & AQP & \subset & B_n \\ & & \cup & & \cup & & \\ Pos & \subset & SQP = B_n^+ & \subset & ASQP & & \end{array}$$

3D

allow one a_{ij}^{-1}

〃 $w\sigma_i^{-1}w^{-1}$

Filtration of B_n

$$\begin{array}{ccccc} AQP & \subset & 2^{nd}AQP & \subset & \cdots \\ \cup & & \cup & & \\ ASQP & \subset & 2^{nd}ASQP & \subset & \cdots \end{array}$$

$B_n = \bigcup_{n=1}^{\infty} AQP_n$

Thm

β is SQP $\iff$ $\inf(\beta) \geqq 0$

β is ASQP $\iff$ $\inf(\beta) = -1$ & $\exists i$ s.t. $\|A_i\| = n-2$.

Def $n(\beta) \overset{df}{=}$ the neg. band #

$= \min \left\{ \#\text{ of } (-) \text{bands in } W \;\middle|\; \begin{array}{l} W \text{ is a word in} \\ BdGen(B_n), \\ W = \beta \end{array} \right\}$

Rmk

$SQP = \{\beta \mid n(\beta) = 0\}$

$ASQP = \{\beta \mid n(\beta) \leq 1\}$

Bennequin inequality.

$$SL(K) \le 2\tau(K) - 1 \lneq 2g_4(K) - 1 \le 2g(K) - 1$$

Contact geom, Heeg Floer, 4D, 3D

$$SL(K) \lneq s(K) - 1 \lneq 2g_4(K) - 1$$

Khovanov

$$\text{Defect} \overset{\text{df}}{=} \frac{(2g(K)-1) - SL(K)}{2} \le n(\beta) \qquad K = \hat{\beta}.$$

conj "=" (Ito-K)

Thm if $\inf(\beta) \le 0$

$$|\inf(\beta)| \le n(\beta) \le (n-2)|\inf(\beta)| - \min\{0, \sup(\beta)\}$$

"=" when $n = 3$